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THE USE OF ARRAY ALGEBRA IN TERRAIN MODELING PROCEDURES.(U)

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THE USE OF ARRAY ALGEBRA IN TERRAIN MODELING PROCEDURES

OCTOBER 1976

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Array algebra, a new technique that finds the least-squares fit of a model equation to a set of ordered data, is investigated as a possible replacement for the current conventional least-squares polynomial-fitting technique. Both techniques are described and analyzed within the context of their applicability to current terrain modeling procedures and are compared for computational efficiency. This analysis specifically considers increasing the number of model polynomial terms (up to 256 terms) and discusses the impact of array algebra upon future R&D activities involving these high order polynomials.

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PREFACE

This report was prepared under the supervision of Mr. W. Howard Carr, Chief, Automated Cartography Branch, Mapping Developments Division, U.S. Army Engineer Topographic Laboratories (ETL).

The author wishes to acknowledge the technical advice and assistance received from Mr. James R. Jancaitis, Automated Cartography Branch. Mr. Jancaitis' publications concerning digital terrain modeling techniques are the basis for many of the observations in this report.

The author further acknowledges the technical advice and assistance received from Mr. William R. Moore, Automated Cartography Branch, and Mr. Michael A. Crombie, Advanced Technology Division, Computer Sciences Laboratory, ETL.

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THE USE OF ARRAY ALGEBRA IN TERRAIN MODELING PROCEDURES

INTRODUCTION. Terrain modeling is currently accomplished by a series of polynomial least-squares fits to observed elevation data. The polynomials are then mathematically joined to provide a second series of polynomials that join smoothly with their neighbors. These final polynomials are then used to generate data for computer-driven plotting devices.

The routine that produces the initial set of polynomials from the observation data utilizes the principle that the best mathematical fit of a model equation to arbitrary points in space must minimize the square of the vertical distance from the data to the surface described by the derived function. This technique involves the inversion of an m by m matrix, where m is equal to the number of terms in the model equation. Although work at ETL currently uses a relatively small (four terms) model equation, new developments will soon require more complexity, expanding the model certainly to nine terms and probably more. Consequently, the inversion time for the m by m matrix will grow geometrically (see appendix D). As a means of reducing some of these cumbersome and time-consuming inversions, the technique of array algebra is being investigated. Array algebra, the invention of Dr. Urho A. Rauhala, performs the same functions as a least-squares fit in fewer computational steps. Where least-squares requires the inversion of one m by m matrix, for instance, array algebra requires the inversion of an a by a matrix and a b by b matrix, where a and b are integer factors of m. As is demonstrated in appendix D, the computational savings of array algebra increase dramatically with m.

This report analyzes the methods by which array algebra can be implemented into ETL's terrain modeling procedure, and determines its feasibility.

ARRAY ALGEBRA IN TERRAIN MODELING.

The Terrain Modeling Procedure. Prior to making any determinations regarding the feasibility of array algebra's implementation, it is necessary to understand the current terrain modeling procedure, which can be ordered into five distinct steps.

1. Data Collection - This step normally involves the analysis of stereo photography of land defined by one topographic map sheet scaled 1:50,000. This analysis yields 2.25 million elevation observations. These observations, however, are from stereo pairs of overlapping photographs and are organized accordingly.

2. Mosaic Routine - This procedure uses a software routine to transform step 1 data into a set of 1.02 million elevations on a single plane congruent to the corresponding 1:50,000 map sheet.

3. Preliminary Least-Squares (L-S) Fits - This step establishes p number of congruent, overlapping rectangles across the data grid and derives a series of fitted L-S polynomials, one for each rectangle. The surfaces defined by these functions, however, do not necessarily join smoothly where they overlap.

4. Interpolation - This step uses the first set of polynomials and a set of weighting functions to produce a second set of polynomials which defines a set of surfaces that exhibit first order continuity (i.e. functions that agree in slope at the points where they meet). This set of surfaces now defines a continuous, smooth-flowing representation of the corresponding terrain.

5. Evaluation of the Final Polynomials - This step involves the evaluation of the final polynomials for each (x, y) at which elevation data is required. These elevations are then used to produce contour maps, DTM data bases, 3-dimensional projections, or any of a variety of topographic products.

Array algebra can be incorporated into this 5-step procedure only in step 3, which currently uses a conventional least-squares technique, and in step 5, which currently uses a standard algebraic solution process. The remainder of this report, therefore, will discuss only these two steps.

Deriving the L-S Polynomials. Each L-S polynomial essentially converts a series of data points into a continuous surface representation. The best representation exists when the polynomials minimize the sum of the squares of the vertical distance from the observations to the defined surface. If the fitted altitude $f(x,y)$ is defined as,

$$f(x_i,y_i) = c_0 + c_1x_i + c_2y_i + c_3x_iy_i$$

then the vertical distance d may be represented as

$$d_i = z_i - f(x_i, y_i)$$

where z_i is the "i"th observation. The entire set of n number of distances can then be represented as

$$\sum_{i=1}^{n} d_i = \sum_{i=1}^{n} (z_i - c_0 - c_1x_i - c_2y_i - c_3x_iy_i)$$

and the sum of the squares of these distances as

$$\sum_{i=1}^{n} d_i^2 = \sum_{i=1}^{n} (z_i - c_0 - c_1x_i - c_2y_i - c_3x_iy_i)^2$$

Minimization of $\sum_{i=1}^{n} d_i^2$ yields, in matrix notation,

$$C_L = (A^TA)^{-1}A^TZ_L$$

where Z_L is an n by 1 matrix of observations, A is an n by m matrix of polynomial terms, and C_L is an m by 1 matrix of coefficients. (A complete derivation and a numerical example are presented in appendix A.

The array algebra method of analysis follows a different approach. Assuming that the elevation data points are on an orthonormal grid (ordered via an x/y grid), then we select an e by f matrix (Z_A) of n observations on the grid and attempt an array algebra polynomial fit. Then, in matrix notation

$$Z_A = XC_AY^T \tag{1}$$

where X is an e by a matrix of x-direction parameters, y is an f by b matrix of y-direction parameters, and C_A is an a by b matrix of coefficients. If we use the same m-term model equation that was used in the least-squares derivation, then a and b must be integer factors of m. Solving (1) for C_A

$$C_A = (X^TX)^{-1}X^TZ_AY(Y^TY)^{-1}$$

(A complete derivation and a numerical example are presented in appendix B).

It may sometimes be desirable to choose ab = m such that a ≠ b. Such a choice increases the accuracy of the fit along the directional parameter which corresponds with the greater factor, while denigrating the fit along the other axis, thus producing a terrain description with predictable unidirectional distortion. This may be extremely useful in saving computational time when analyzing terrain that exhibits a unidirectional trend across a specified area.

The complexity and extreme length of Dr. Rauhala's array algebra derivation has impeded investigations into its validity, a fact that has hindered its acceptance even in the face of a high correlation of empirical data, and there were those who would not accept the equivalence of

$$(A^TA)^{-1}A^TZ_L \qquad \text{and} \qquad (X^TX)^{-1}X^TZ_AY(Y^TY)^{-1}.$$

However, Mr. James R. Jancaitis, Automated Cartography Branch, ETL, recently constructed a detailed proof[1] of the equivalence of the array algebra solution to that of least-squares, and also developed techniques for weighting and constraining the array algebra fit.

Impact Upon the Terrain Modeling Procedure. An observation of the numerical examples in appendixes A and B, where identical best fits are determined for the same data using both the least-squares and array algebra techniques, coupled with Mr. Jancaitis' proof of least-squares array algebra equivalency, justifies the statement that the implementation of array algebra into ETL's current modeling software would have no impact upon the resulting terrain model. The only observable difference would deal with the computational efficiency of the polynomial fitting software.

As best as can be predicted, therefore, array algebra is a feasible method for fitting a model equation to a set of orthonormally ordered data.

AN ANALYSIS OF THE COMPUTATIONAL EFFICIENCY OF ARRAY ALGEBRA.

The Dual Nature of the Polynomial Fitting Process. To measure the relative efficiency of array algebra, the computational differences between the least-squares and array algebra techniques must be isolated. This requires a quantitative comparison of the number of multiplications and additions needed to compute the coefficient matrices C_L or C_A when

$$C_L = (A^TA)^{-1}A^T \ Z_L$$

or

$$C_A = (X^TX)^{-1}X^T \ Z_A \ Y(Y^TY)^{-1}$$

---

[1]James R. Jancaitis, "Theroetical Analysis of Array Algebra," Paper Available at USA Engineer Topographic Laboratories, Fort Belvoir, VA, September 1976.

An important feature of this analysis is the repetitive aspects of these computations. Since each C_L or C_A describes only one polynomial, then the polynomial must be calculated p number of times to describe an entire 1:50,000 map sheet. The data presented in Mr. Jancaitis' report[2] leads to the derivation of p as

$$p = \left[\text{INTEGER ROUNDED UP } \frac{916}{\frac{1}{2}(e+1)}\right] \left[\text{INTEGER ROUNDED UP } \frac{1112}{\frac{1}{2}(f+1)}\right]$$

where e & f are the respective number of points on the sides of each rectangle of data. It should be noted that p is subject to variations, since the number of observations on a 1:50,000 map sheet often vary considerably from the example.

At first glance it seems that the number of multiplications and additions needed to compute the coefficient matrix must be multiplied by p to predict the number of computations per map sheet. However, this is not true. The computation of $(A^TA)^{-1}A^T$ essentually involves only the matrix A, which is composed of m model equation terms evaluated at a given (x,y), listed over all (x,y) considered for each fit. The values for A, therefore, are dependent only upon the model equation and the position of the observations with respect to the origin. If a new origin is selected for each computation of C_L and if it is placed in the same relative position with respect to the new set of observations, then A is constant throughout the entire polynomial fitting process. Thus, the computation of $(A^TA)^{-1}A^T$ is a one-time procedure, and only the multiplication of $(A^TA)^{-1}A^T$ to Z_L needs to be repeated p times. A similar procedure is used with the array algebra format, showing that the computation of $(X^TX)^{-1}X^T$ and $Y(Y^TY)^{-1}$ are one-time calculations while the multiplication of $(X^TX)^{-1}X^T$ to Z_L and the multiplication of $(X^TX)^{-1}X^TZ_L$ to $Y(Y^TY)^{-1}$ must be repeated p times per map sheet.

An analysis of multiplications and additions required to fit a model equation of m terms to a series of rectangular data grids of n points yields the following figures (see appendix C for a complete derivation):

---

[2] James R. Jancaitis, "Modeling and Contouring Irregular Surfaces Subject to Constraints," USA Engineer Topographic Laboratories, Fort Belvoir, VA, ETL-CR-74-19, AD A010406, January 1975.

Where $C_L = (A^TA)^{-1}A^T \quad Z_L$

C_L requires:

$1/6m(9mn + 3n + 4m^2 + 3m - 1)$ one-time multiplications

mnp repetitive multiplications

$1/6m(9mn - 3n + 4m^2 - 6m - 4)$ one-time additions

$mp(n-1)$ repetitive additions

Where $C_A = (X^TX)^{-1}X^T \quad Z_A \quad Y(Y^TY)^{-1}$ and $ab = m$

C_A requires:

$1/6[a(9a\sqrt{n}+3\sqrt{n}+4a^2+3a-1) + b(9b\sqrt{n}+3\sqrt{n}+4b^2+b-1)]$ one-time multiplications

$(an + ab\sqrt{n})p$ repetitive multiplications

$1/6[a(9a\sqrt{n}-3\sqrt{n}+4a^2-6a-4) + b(9b-\sqrt{n}-3-\sqrt{n}+4b^2-6b-4)]$ one-time additions

$(an - a\sqrt{n} + m\sqrt{n}-m)p$ repetitive additions

There are some situations in which some computational steps may be saved. For an explanation of these cases, refer to appendix C.

A Comparison of the Computational Efficiency of Least-Squares to that of Array Algebra.

An Analysis According to the Complexity of the Model Equation. An analysis was conducted (see appendix E) to determine the computational savings of array algebra as the model equation increased in complexity from 4 to 256 terms. To simulate undistorted terrain modeling, the array algebra parameters selected were $a = b = \sqrt{m}$, where m was a perfect square, and $a < b$ with a and b as near m as possible when m was not a perfect square. The square observation grid was selected over a range of n, compatible with the complexity of the model equation, as

determined by Mr. Jancaitis.[3] The results of this analysis show that the advantage of array algebra's smaller matrix inversions is diminished by the fact that the inversion process is a one-time procedure. The number of repetitive multiplications for the conventional least-squares procedure, however, exceeds those of the array algebra procedure by

$$
\begin{aligned}
& mpn - ap(n + b\sqrt{n}) \\
&= mpn - apn - apb\sqrt{n} \\
&= mpn - apn - mp\sqrt{n} \\
&= mpn\left(1 - \frac{1}{b} - \frac{1}{\sqrt{m}}\right)
\end{aligned}
$$

Since $b \geq 2$ and since $\sqrt{n} \geq 3$, then $mpn\left(1 - \frac{1}{b} - \frac{1}{\sqrt{m}}\right)$ is always positive, and therefore array algebra requires fewer repetitive multiplications than conventional least-squares. Note also that the difference increases with b or $\sqrt{n}$. A similar result is produced with the number of repetitive additions.

Appendix E presents a detailed listing of computer time saved, considering only multiplication and addition as variable factors in the modeling software. Use of a CDC 6400 computer is assumed, with no time-sharing problems. The data from this listing show that under the current terrain modeling parameters used by Mr. Jancaitis[4], where

$m = 4$ terms

$a = b = 2$

$e = f = 13$

$n = 169$ observations

The computer savings using array algebra will amount to only 41.13 seconds per 1:50,000 map sheet. More significant savings, however, can be realized where the model equation is large. Large model equations allow for better terrain resolution over a larger grid, enabling larger values for n to be selected which in turn lowers the number of polynomials required to fit a 1:50,000 map sheet. An attempt to do this using the

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[3] op. cit.
[4] op. cit.

least-squares technique fails because the multiplication and add time become unreasonably large as m and n increase. Array algebra, however, enables the procedure to be conducted without unduly large increases in computer time. Note figure 1, which exhibits the relationship between the length of the model equation and the necessary computer time for the required additions and multiplications, assuming that n remains optimal for each m. That is

$$n = (\text{INTEGER}\sqrt{42.25m})^2$$

which is a relationship that is derived from information in Modeling and Contouring Irregular Surfaces Subject to Constraints[5]. This value for n merely assures a consistent terrain representation by matching the size of terrain represented by one polynomial with a sufficiently large model equation.

An Analysis According to the Array Algebra Parameters. The multiplication of

$$(X^TX)^{-1}X^T \quad \text{to} \quad Z_A \quad \text{to} \quad Y(Y^TY)^{-1}$$

requires $an+mf$ multiplications and $an + af + mf - m$ additions (see appendix C). Since $ab = m$ and $ef = n$, the number of multiplications and additions increase as a is increased or as f is increased. As a result, the optimal values for a and b, i.e. those values that minimize the number of multiplications and additions, are $a = 1$ and $b = m$. Likewise, e and f seem optimal at $e = m$ and $f = 1$. Remember, however, that a and b define the number of terms in the model equation that carries x or y values and that as a and b deviate from $a = b=\sqrt{m}$ the model equation becomes x-oriented or y-oriented, producing a commensurate lateral distortion of the fitted terrain representation. A similar situation develops as e and f deviate from $e = f=\sqrt{n}$. Since e and f define the length of the sides of the fitted rectangular grids, these grids deviate from squares to thin rectangles. Since the length of either the x or the y direction increases, there must be a corresponding increase in the x or y terms of the model equation to describe the extended terrain area. Thus, any variance of e must be accompanied by a corresponding variance in a, which leads to the unidirection distortion described above.

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[5]op. cit.

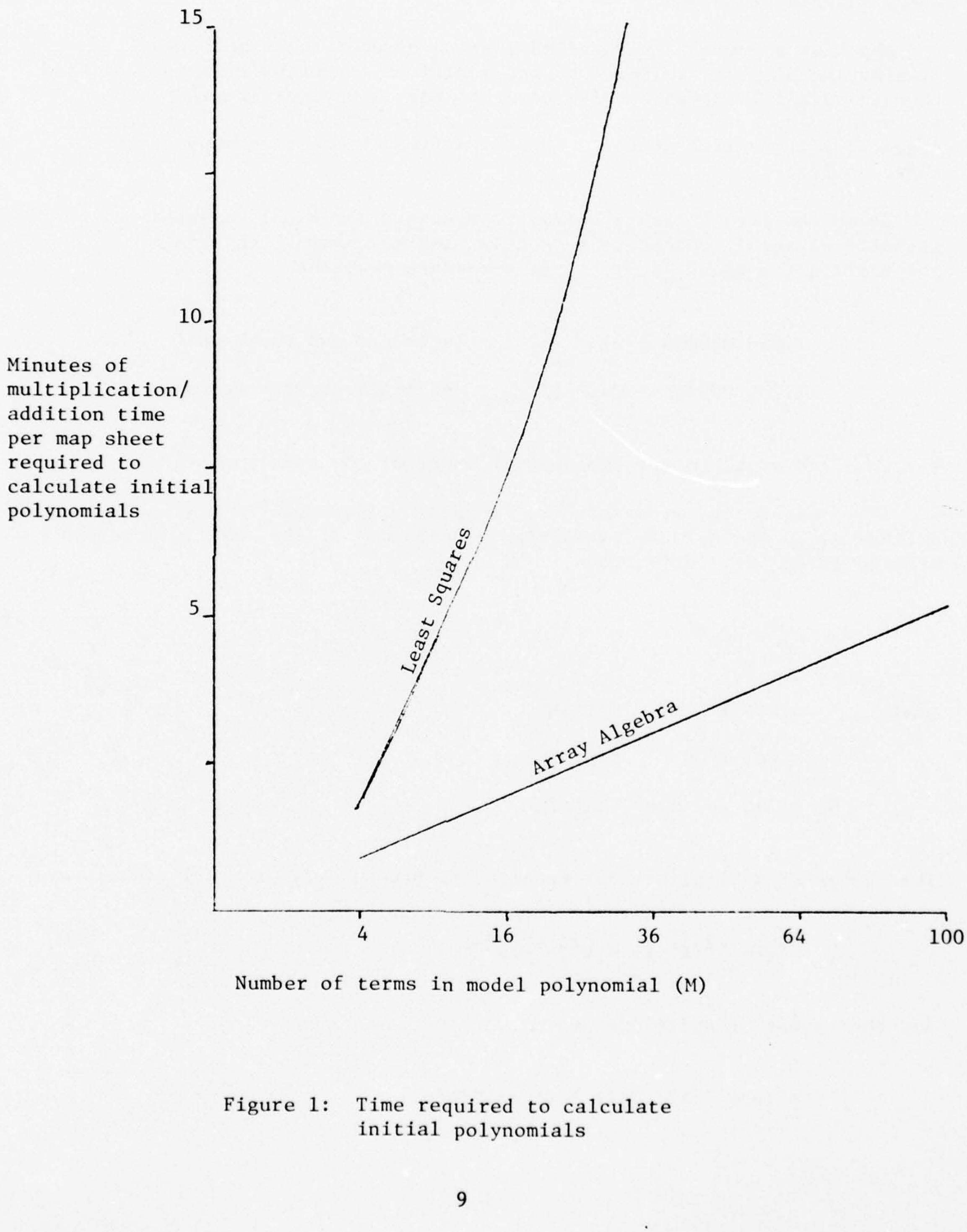

Figure 1: Time required to calculate initial polynomials

A quantitative analysis of this distortion can be performed empirically if the present modeling software is altered to incorporate the array algebra technique.

Appendix F presents a detailed listing of computer time saved, considering only multiplications and additions as variable factors on the terrain modeling software, with several model equations studied for differing values of a and b. Use of a CDC 6400 computer is still assumed, with no time sharing. The data from the listing supports the above analysis.

Using the Array Algebra Format to Evaluate the Final Polynomials. A standard algebraic method is currently used to evaluate the final polynomials for each (x,y). This procedure requires

$1{,}018{,}000q(q + 2)$	adds per map sheet and
$2{,}036{,}000(q^2 + 2q - 1)$	multiplications per map sheet,

where q is equal to the exponential order of the model equation.

If, however, the array algebra x and y parameter matrices are defined as in the polynomial fitting process and if the coefficients are ordered in an x-y grid, then

$$Z_A = XC_AY^T$$

where X is an $e' \times a'$ matrix

Y is an $f' \times b'$ matrix

C_A is an $a' \times b'$ matrix

The number of multiplications required to produce Z_A for all (x,y) are

$$p'[(a')^2(e'-1) + b'(e'-1)^2]$$

and the number of additions are

$$p'[a'(e'-1)(a'-1) + (e'-1)^2(b'-1)]$$

where, as derived from Modeling and Contouring Irregular Surfaces Subject to Constraints[6]

$$p' = \left(\text{INTEGER ROUNDED UP } \frac{916}{\frac{1}{2}(e'+1)}\right)\left(\text{INTEGER ROUNDED UP } \frac{1112}{\frac{1}{2}(f'+1)}\right)$$

A detailed examination of these formulas is presented in appendix G.

The computer times for the multiplications and additions necessary to evaluate all of the polynomials over an entire map sheet were analyzed, and a comparison of conventional versus array algebra methods was conducted. The conventional system indicates a considerable rate of increase in computational computer time as the length of the model equation is increased, a fact that has contributed to the reluctance to use higher order polynomials in terrain modeling. The array algebra format, however, enables high order polynomials to be used without significantly increasing the evaluation time. The multiplication and addition time, for instance, for a 4th order polynomial evaluation over all (x,y) is 56.19 seconds and for an 18th order polynomial is 168.84 seconds. As a comparison, the corresponding conventional times are 26.59 seconds and 3,235.13 seconds. The following graph (figure 2) indicates the results of the analysis.

DISCUSSION. A major roadblock in using higher order polynomials for terrain modeling has been the unreasonable increase in required computer time. Indeed, the change from a 1st order to a 15th order model equation increases the computer time over all five terrain modeling steps by almost five hours per map sheet. A comparative change using array algebra results in an increase of only 9 3/4 minutes. It should be noted, however, that high order polynomials may have drawbacks that are unrelated to the increase in computer time. Array algebra, by allowing these drawbacks to be investigated without undue computer costs, provides an option that is not reasonably available with conventional least-squares.

When the use of high order polynomials is investigated, the software required will be lengthy and exceedingly intricate. Since conventional least-squares cannot handle these polynomials within reasonable computer times, array algebra must be used. The investigation of computer use of high order polynomials and array algebra simultaneously would be excessively complex since no workable basic structure exists for either development. Plans at the US Army Engineer Topographic Laboratories, however, call for the current terrain modeling software (1st order polynomials) to be revised to incorporate an array algebra solution technique, followed by efforts to

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[6] op. cit.

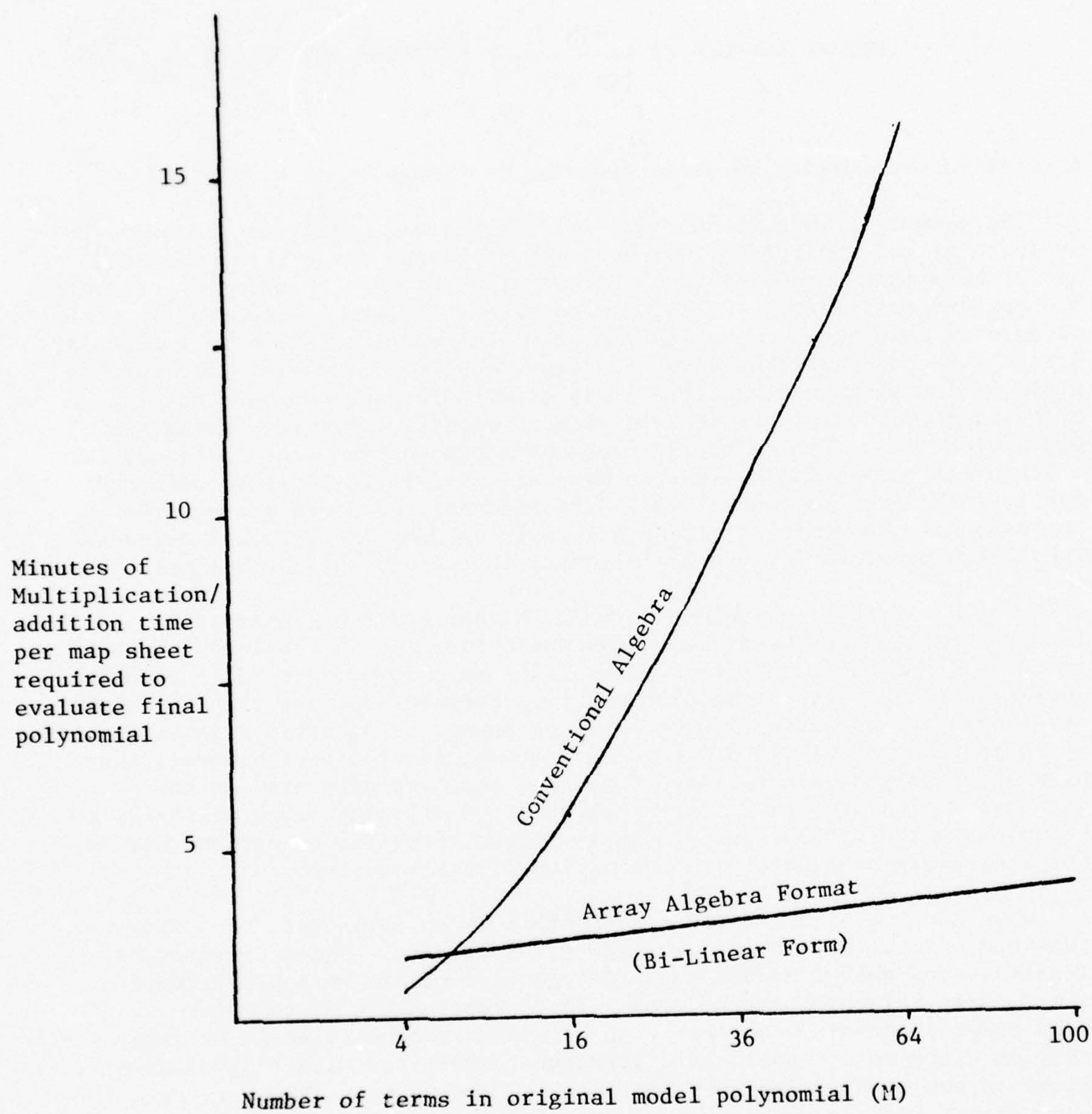

Figure 2. Time required to evaluate final polynomials

expand the model equation. Thus, the investigation of high order polynomials will have array algebra available as a tool rather than as a problem.

With the advent of the array algebra solution technique and especially in light of the recent work by Jancaitis, the conventional least-squares software has become at least partially obsolete for this application. Since the efficient use of computer time is becoming increasingly important in virtually all aspects of automated cartography, array algebra may prove to be a powerful tool in the search for cost-effective digital terrain modeling techniques.

CONCLUSIONS. It is concluded that:

1. Array algebra produces the same results as the conventional least-squares method.[7]

2. Array algebra can be weighted and constrained.[7]

3. Array algebra performs a least-squares type polynomial fit faster than the conventional least-squares method.

4. Acceptance of either x-direction or y-direction lateral distortion decreases array algebra computational time commensurate with the degree of distortion.

5. Array algebra evaluates the final polynomial faster than the conventional algebraic method in all cases save that of the smallest possible model equation where array algebra is 30 seconds slower per map sheet.

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[7] James R. Jancaitis, "Theoretical Analysis of Array Algebra," paper available at USA Engineer Topographic Laboratories, Fort Belvoir, VA, September 1976.

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APPENDIX A. LEAST-SQUARES POLYNOMIAL FITTING

THE DERIVATION OF $C_L = (A^T A)^{-1} A^T Z_L$

Given a set of n elevation observations, $\sum_{i=1}^{n} z_i$, suppose that one wishes to fit the polynomial

$$f(x_i, y_i) = c_0 + c_1 x_i + c_2 y_i + c_3 x_i y_i$$

as closely as possible to these observations. An accepted "best" fit occurs when the sum of the squares of the vertical distances from the observations to the described surface is minimized. The distance from an elevation point to the described surface may be defined as

$$d = z_i - f(x_i, y_i)$$

$$d^2 = (z_i - f(x_i, y_i))^2$$

The set of n distance-squares is therefore

$$S = \sum_{i=1}^{n} d_i^2 = \sum_{i=1}^{n} (z_i - c_0 - c_1 x_i - c_2 y_i - c_3 x_i y_i)^2$$

S will be minimized when its derivative is equal to zero.

$$
\left.
\begin{aligned}
{}^{*}\ \frac{\delta S}{\delta c_0} &= -2\Sigma(z_i - c_0 - c_1x_i - c_2y_i - c_3x_iy_i) = 0 \\
\frac{\delta S}{\delta c_1} &= -2\Sigma[(z_i - c_0 - c_1x_i - c_2y_i - c_3x_iy_i)(x_i)] = 0 \\
\frac{\delta S}{\delta c_2} &= -2\Sigma[(z_i - c_0 - c_1x_i - c_2y_i - c_3x_iy_i)(y_i)] =0 \\
\frac{\delta S}{\delta c_3} &= -2\Sigma[(z_i - c_0 - c_1x_i - c_2y_i - c_3x_iy_i)(x_iy_i)] = 0
\end{aligned}
\right\} \quad (A1)
$$

$$
\left.
\begin{aligned}
&c_0n + c_1 \Sigma x_i + c_2 \Sigma y_i + c_3 \Sigma x_iy_i = \Sigma z_i \\
&c_0\Sigma x_i + c_1 \Sigma x_i^2 + c_2 \Sigma x_iy_i + c_3 \Sigma x_i^2y_i = \Sigma x_iz_i \\
&c_0 \Sigma y_i + c_1 \Sigma x_i y_i + c_2 \Sigma y_i^2 + c_3 \Sigma x_i y_i^2 = \Sigma y_iz_i \\
&c_0 \Sigma x_iy_i + c_1 \Sigma x_i^2y_i + c_2 \Sigma x y_i^2 + c_3 \Sigma x_i^2y_i^2 = \Sigma x_iy_iz_i
\end{aligned}
\right\} \quad (A2)
$$

---

*Hereafter the symbol Σ indicates $\sum\limits_{i=1}^{n}$

Define A as an n by m matrix of polynomial terms, C_L as an m by 1 matrix of coefficients, and Z_L as an n by 1 matrix of observations.

Then,

$$A = \begin{bmatrix} 1 & x_1 & y_1 & x_1y_1 \\ 1 & x_2 & y_2 & x_2y_2 \\ 1 & x_3 & y_3 & x_3y_3 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & y_n & x_ny_n \end{bmatrix}$$

$$C_L = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$Z_L \quad \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_n \end{bmatrix}$$

The left side of equation (A2) can now be written as

$$\begin{bmatrix} 1 & 1 & 1 \ldots & 1 \\ x_1 & x_2 & x_3 \ldots & x_n \\ y_1 & y_2 & y_3 \ldots & y_n \\ x_1y_1 & x_2y_2 & x_3y_3 \ldots & x_ny_n \end{bmatrix} \begin{bmatrix} 1 & x_1 & y_1 & x_1y_1 \\ 1 & x_2 & y_2 & x_2y_2 \\ 1 & x_3 & y_3 & x_3y_3 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & y_n & x_ny_n \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

which in matrix notation is

$$A^TAC_L$$

The right side of equation (A2) can now be written as

$$\begin{bmatrix} 1 & 1 & 1 \ldots & 1 \\ x_1 & x_2 & x_3 \ldots & x_n \\ y_1 & y_2 & y_3 \ldots & y_n \\ x_1y_1 & x_2y_2 & x_3y_3 \ldots & x_ny_n \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_n \end{bmatrix}$$

which in matrix notation is

$$A^TZ_L$$

Equation (A2) can now be rewritten as

$$A^T A C_L = A^T Z_L$$

Solving for C_L , multiply the left side by $(A^T A)^{-1}$

$$(A^T A)^{-1}(A^T A)C_L = (A^T A)^{-1} A^T Z_L$$

$$C_L = (A^T A)^{-1} A^T Z_L$$

A NUMERICAL EXAMPLE OF THE LEAST-SQUARES TECHNIQUE. Given nine observations on a 3 by 3 grid,

$z(0,0) = 100$

$z(0,1) = 110$

$z(0,2) = 112$

$z(1,0) = 118$

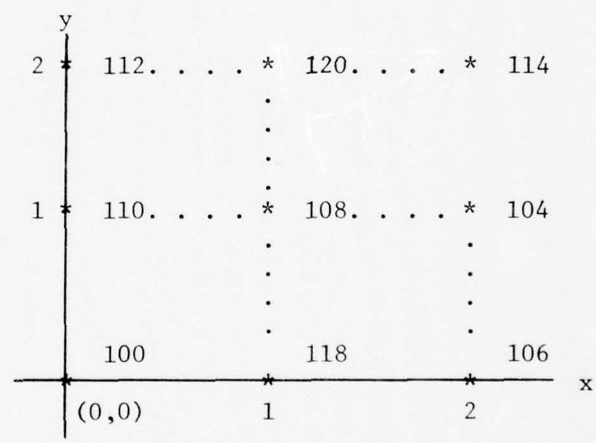
$z(1,1) = 108$

$z(1,2) = 120$

$z(2,0) = 106$

$z(2,1) = 104$

$z(2,2) = 114$



The model equation is chosen as

$$f(x,y) = c_0 + c_1 x + c_x y + c_3 xy$$

Then:

$$Z_L = \begin{bmatrix} 100 \\ 110 \\ 112 \\ 118 \\ 108 \\ 120 \\ 106 \\ 104 \\ 114 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 4 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 9 & 9 & 9 & 9 \\ 9 & 15 & 9 & 15 \\ 9 & 9 & 15 & 15 \\ 9 & 15 & 15 & 25 \end{bmatrix}$$

$$(A^T A)^{-1} = \frac{1}{36} \begin{bmatrix} 25 & -15 & -15 & 9 \\ -15 & 15 & 9 & -9 \\ -15 & 9 & 15 & -9 \\ 9 & -9 & -9 & 9 \end{bmatrix}$$

$$(A^T A)^{-1} A^T = \frac{1}{36} \begin{bmatrix} 25 & 10 & -5 & 10 & 4 & -2 & -5 & -2 & 1 \\ -15 & -6 & 3 & 0 & 0 & 0 & 15 & 6 & -3 \\ -15 & 0 & 15 & -6 & 0 & 6 & 3 & 0 & -3 \\ 9 & 0 & -9 & 0 & 0 & 0 & -9 & 0 & 9 \end{bmatrix}$$

$$(A^TA)^{-1}A^TZ_L = \frac{1}{36}\begin{bmatrix} 3788 \\ 48 \\ 168 \\ -36 \end{bmatrix} = \begin{bmatrix} 105.222 \\ 1.333 \\ 4.667 \\ -1.000 \end{bmatrix}$$

$c_0 = 105.222 \qquad c_1 = 1\ 1/3 \qquad c_2 = 4\ 2/3 \qquad c_3 = -1$

$$z = 105.222 + 1.333x + 4.667y - xy$$

Notice that this result is identical with that of the array algebra numerical example in appendix B.

APPENDIX B. ARRAY ALGEBRA POLYNOMIAL FITTING

THE DERIVATION OF $C_A = (X^T X)^{-1} X^T Z_A Y (Y^T Y)^{-1}$

When the set of elevation observations

$$\sum_{i=0}^{m-1} z_i$$

is ordered in an orthonormal grid, then it may be written as

$$\sum_{i,j=0}^{(a-1),(b-1)} z_{ij}$$

with ab=m terms in the model equation. If the model equation is chosen as the four-term polynomial

$$z_{ij} = (x_i, y_i) = c_{00} + c_{10} x_i + c_{01} y_i + c_{11} x_i y_i$$

then,

$$z_{ij} = \sum_{k,l=0}^{1,1} \left(c_{kl} x_i^k y_j^l \right)$$

$$= \begin{bmatrix} 1 & x_i \end{bmatrix} \begin{bmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{bmatrix} \begin{bmatrix} 1 \\ y_j \end{bmatrix}$$

and, given $n=ef$ observations, then all z_{ij} in the grid are

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_e \end{bmatrix} \begin{bmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{bmatrix} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ y_1 & y_2 & \cdots & y_f \end{bmatrix}$$

$$= Z_A = XC_AY^T$$

where

$$X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_e \end{bmatrix}$$

$$C_A = \begin{bmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{bmatrix}$$

$$Y = \begin{bmatrix} 1 & y_1 \\ 1 & y_2 \\ \vdots & \vdots \\ 1 & y_f \end{bmatrix}$$

It is, of course, possible to generalize this derivation by selecting a general model polynomial.

Thus, choosing

$$z_{ij}=f(x_i y_j)=c_{00}+c_{10}x_i+c_{01}y_j+c_{11}x_i y_j+c_{20}x_i^2+c_{02}y_j^2$$

$$+c_{21}x_i^2 y_j+c_{12}x_i y_j^2+c_{22}x_i^2 y_j^2+\ldots+c_{(a-1)(b-1)}x_i^{(a-1)}y_j^{(b-1)}$$

then,

$$z_{ij}=\sum_{k,l=0}^{(a-1),(b-1)} c_{kl}x_i^k y_j^l$$

$$=\begin{bmatrix} 1 & x_i & x_i^2 & x_i^3 \ldots x_i^{(a-1)} \end{bmatrix}\begin{bmatrix} c_{00} & c_{01} & c_{02} & \cdots & c_{0(b-1)} \\ c_{10} & c_{11} & c_{12} & \cdots & c_{1(b-1)} \\ c_{20} & c_{21} & c_{22} & \cdots & c_{2(b-1)} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ c_{(a-1)0} & c_{(a-1)1} & c_{(a-1)2} & \cdots & c_{(a-1)(b-1)} \end{bmatrix}\begin{bmatrix} 1 \\ y_j \\ y_j^2 \\ \cdot \\ \cdot \\ \cdot \\ y_j^{(b-1)} \end{bmatrix}$$

and all z_{ij} in the grid are

$$= \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{(a-1)} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{(a-1)} \\ 1 & x_3 & x_3^3 & \cdots & x_3^{(a-1)} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 1 & x_e & x_e^2 & \cdots & x_e^{(a-1)} \end{bmatrix} \begin{bmatrix} c_{00} & c_{01} & c_{02} & \cdots & c_{0(b-1)} \\ c_{10} & c_{11} & c_{12} & \cdots & c_{1(b-1)} \\ c_{20} & c_{21} & c_{22} & \cdots & c_{2(b-1)} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ c_{(a-1)0} & c_{(a-1)1} & c_{(a-1)2} & \cdots & c_{(a-1)(b-1)} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ y_1 & y_2 & y_3 & \cdots & y_f \\ y_1^2 & y_2^2 & y_3^2 & \cdots & y_f^2 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ y_1^{(b-1)} & y_2^{(b-1)} & y_3^{(b-1)} & \cdots & y_f^{(b-1)} \end{bmatrix}$$

$$= Z_A = XC_AY^T$$

Where X is a e by a matrix of x-direction parameters, Y is a f by b matrix of y-direction parameters, C_A is an a by b matrix of coefficients, and Z_A is a e by f matrix of elevation observations.

Solving for C_A

$$Z_A = XC_AY^T$$

$$Z_AY = X\,C_AY^TY$$

$$X^TZ_AY = (X^TX)C_A(Y^TY)$$

$$(X^TX)^{-1}\,X^TZ_AY = (X^TX)^{-1}(X^TX)C_A(Y^TY) = C_A(Y^TY)$$

$$(X^TX)^{-1}X^TZ_AY(Y^TY)^{-1} = C_A(Y^TY)(Y^TY)^{-1} = C_A$$

$$\underline{C_A = (X^TX)^{-1}X^TZ_AY(Y^TY)^{-1}}$$

Note that when $X = Y$, a special computational case exists:

$$Y(Y^TY)^{-1} = X(X^TX)^{-1}$$

$$= \left[[X({}^TX)^{-1}]^T \right]^T$$

$$= \left[[(X^TX)^{-1}]^T X^T \right]^T$$

and, since $(X^TX)^{-1}$ is symmetric,

$$\left[[(X^TX)^{-1}]^T X^T \right]^T = [(X^TX)^{-1}X^T]^T$$

therefore, when $X = Y$,

$$C_A = [(X^TX)^{-1} X^T] \, Z_A \, [(X^TX)^{-1} X^T]^T$$

and $Y(Y^TY)^{-1}$ need not be calculated.

A NUMERICAL EXAMPLE OF THE ARRAY ALGEBRA TECHNIQUE. The same observational values used in the numerical example in appendix A will also apply to this example.

The model equation $z = c_{00} + c_{01}x + c_{10}y + c_{11}xy$ can be written

$$z = (1,x) \begin{bmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{bmatrix} \begin{bmatrix} 1 \\ y \end{bmatrix}$$

therefore,

$$\begin{bmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{bmatrix} = (X^TX)^{-1} X^T Z_A Y(Y^TY)^{-1}$$

where $X = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$ $\quad Y = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$

and $$Z_A = \begin{bmatrix} z_{00} & z_{01} & z_{02} \\ z_{10} & z_{11} & z_{12} \\ z_{20} & z_{21} & z_{22} \end{bmatrix} = \begin{bmatrix} 100 & 110 & 112 \\ 118 & 108 & 120 \\ 106 & 104 & 114 \end{bmatrix}$$

$$(X^TX) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}$$

$$(X^TX)^{-1} = \frac{1}{6} \begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix}$$

$$(X^TX)^{-1}X^T = \frac{1}{6} \begin{bmatrix} 5 & -3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 5 & 2 & -1 \\ -3 & 0 & 3 \end{bmatrix}$$

since $Y = X$, $Y(Y^TY)^{-1} = [(X^TX)^{-1}X^T]^T$

$$Y(Y^TY)^{-1} = \frac{1}{6} \begin{bmatrix} 5 & -3 \\ 2 & 0 \\ -1 & 3 \end{bmatrix}$$

$$C_A = \frac{1}{6} \begin{bmatrix} 5 & 2 & -1 \\ -3 & 0 & 3 \end{bmatrix} \begin{bmatrix} 100 & 110 & 112 \\ 118 & 108 & 120 \\ 106 & 104 & 114 \end{bmatrix} \frac{1}{6} \begin{bmatrix} 5 & -3 \\ 2 & 0 \\ -1 & 3 \end{bmatrix}$$

$$= \frac{1}{36} \begin{bmatrix} 630 & 662 & 686 \\ 18 & -18 & 6 \end{bmatrix} \begin{bmatrix} 5 & -3 \\ 2 & 0 \\ -1 & 3 \end{bmatrix}$$

$$= \frac{1}{36} \begin{bmatrix} 3788 & 168 \\ 48 & -36 \end{bmatrix}$$

$$= \begin{bmatrix} 105.222 & 4\ 2/3 \\ 4/3 & -1 \end{bmatrix}$$

$$Z = 105.222 + 1.333x + 4.667y - xy$$

Notice that this result is identical with that of the least-squares numerical example in appendix A.

APPENDIX C. THE NUMBER OF MULTIPLICATIONS AND ADDITIONS REQUIRED TO FIT A POLYNOMIAL OF m TERMS TO DATA SETS OF n OBSERVATIONS

REQUIRED MULTIPLICATIONS AND ADDITIONS USING THE METHOD OF LEAST-SQUARES.

Given that

$$C_L = (A^TA)^{-1}A^T \; Z_L$$

where

A is an $n\times m$ matrix, Z_L is an $n\times 1$ matrix.

Then

$L = A^TA$ requires

$n\sum_{i=1}^{m} i$ multiplications and $(n-1)\sum_{i=1}^{m} i$ additions

which simplify to

$\frac{1}{2}\,nm(m+1)$ multiplications and $\frac{1}{2}\,m(n-1)(m+1)$ additions.

$P = (L)^{-1}$ requires

$\frac{1}{6}\,m(4m^2+3m-1)$ multiplications (see appendix D)

and

$\frac{1}{6}\,m(4m^2-3m-1)$ additions (see appendix D).

$Q = PA^T$ requires

m^2n multiplications and $mn(m-1)$ additions.

$R = (A^TA)^{-1}A^T$ requires

$\frac{1}{6}m(4m^2 + 3m + 9mn + 3n - 1)$ multiplications

and

$\frac{1}{6}m(4m^2 - 6m + 9mn - 3n - 4)$ additions.

In the polynomial fitting process, it is necessary to compute $(A^TA)^{-1}A^T$ only once. The multiplication of $(A^TA)^{-1}A^T$ to Z_L, however, must be repeated as many times as necessary to compute all of the polynomials needed to describe the terrain.

This process requires

(mn)(# of reps.) multiplications, and

m(n-1)(# of reps.) additions.

REQUIRED MULTIPLICATIONS AND ADDITIONS USING ARRAY ALGEBRA.

$$C_A = (X^TX)^{-1}X^T \quad Z_A \quad Y(Y^TY)^{-1}$$

where

X is an e x a matrix

Y is an f x b matrix

ab = m(terms in the model equation)

ef = n(number of observations)

X^TX requires

$\frac{1}{2}$ ae(a + 1) multiplications

and

$\frac{1}{2}$ a(e - 1)(a + 1) additions.

$(X^TX)^{-1}$ requires

$\frac{1}{6}$ a(4a^2 + 3a - 1) multiplications

and

$\frac{1}{6}$ a(4a^2 - 3a - 1) additions (see appendix D).

$(X^TX)^{-1}$ X^T requires

a^2e multiplications .

and

ae(a - 1) additions.

$(X^TX)^{-1}X^T$ therefore requires

$\frac{1}{6}$ a(4a^2 + 3a + 9ae + 3e - 1) multiplications

and

$$\frac{1}{6}\, a(4a^2 - 6a + 9ae - 3e - 4) \qquad \text{additions.}$$

$Y(Y^TY)^{-1}$ likewise requires

$$\frac{1}{6}\, b(4b^2 + 3b + 9bf + 3f - 1) \qquad \text{multiplications}$$

and

$$\frac{1}{6}\, b(4b^2 - 6b + 9bf - 3f - 4) \qquad \text{additions.}$$

The repetitive computations involve the multiplication of $(X^TX)^{-1}X^T$ to Z and the multiplication of $(X^TX)^{-1}X^TZ$ to $Y(Y^TY)^{-1}$. These steps require

$$p(an + mf) \qquad \text{multiplications}$$

and

$$p(an - af + mf - m) \qquad \text{additions}$$

where p is equal to the number of polynomial fits per map sheet.

Where $X = Y$,

$$Y(Y^TY)^{-1} = X(X^TX)^{-1}$$

$$= \left([X(X^TX)^{-1}]^T\right)^T$$

which, since $(X^TX)^{-1}$ is symmetric, is equal to $[(X^TX)^{-1}X^T]^T$

therefore, $Y(Y^TY)^{-1}$ need not be calculated.

APPENDIX D.. THE NUMBER OF MULTIPLICATIONS AND ADDITIONS NECESSARY TO INVERT AN mxm MATRIX USING GAUSSIAN ELIMINATION

Given that:

$$
A = \left[\begin{array}{ccccc|ccccc}
a_{11} & a_{12} & a_{13} & \cdots & a_{1m} & 1 & 0 & 0 & \cdots & 0 \\
a_{21} & a_{22} & a_{23} & \cdots & a_{2m} & 0 & 1 & 0 & \cdots & 0 \\
a_{31} & a_{32} & a_{33} & \cdots & a_{3m} & 0 & 0 & 1 & \cdots & 0 \\
\cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & & \cdot \\
\cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & & \cdot \\
\cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & & \cdot \\
a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mm} & 0 & 0 & 0 & \cdots & 1
\end{array}\right]
$$

$$
B = \left[\begin{array}{ccccc|ccccc}
1 & b_{12} & b_{13} & \cdots & b_{1m} & X & 0 & 0 & \cdots & 0 \\
0 & b_{22} & b_{23} & \cdots & b_{2m} & X & 1 & 0 & \cdots & 0 \\
0 & b_{32} & b_{33} & \cdots & b_{3m} & X & 0 & 1 & \cdots & 0 \\
\cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & & \cdot \\
\cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & & \cdot \\
\cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & & \cdot \\
0 & b_{m2} & b_{m3} & \cdots & b_{mm} & X & 0 & 0 & \cdots & 1
\end{array}\right]
$$

$$
C = \left[\begin{array}{cccccc|cccccc}
1 & 0 & b_{13} & \cdots & b_{1m} & & X & X & 0 & \cdots & 0 \\
0 & 1 & c_{23} & \cdots & c_{3m} & & X & X & 0 & \cdots & 0 \\
0 & 0 & c_{33} & \cdots & c_{3m} & & X & X & 1 & \cdots & 0 \\
\cdot & \cdot & \cdot & & \cdot & & \cdot & \cdot & \cdot & & \cdot \\
\cdot & \cdot & \cdot & & \cdot & & \cdot & \cdot & \cdot & & \cdot \\
\cdot & \cdot & \cdot & & \cdot & & \cdot & \cdot & \cdot & & \cdot \\
0 & 0 & c_{m3} & \cdots & c_{mm} & & X & X & 0 & \cdots & 1
\end{array}\right]
$$

Gaussian reduction from A to B requires m^2 multiplications and $m(m-1)$ additions. Reduction from B to C requires an identical number of operations. Thus, a complete mxm matrix inversion requires

m^3 multiplications

and

$m^2(m-1)$ additions.

When the original matrix is symmetric, however, procedures are available that reduce the required multiplications and additions by a considerable amount.

Diagonalizing the matrix, we see that reduction from A to B requires, as above,

m^2 multiplications

and

$m(m-1)$ additions.

The next step in the diagonalization involves reducing B to C, where

$$C = \left[\begin{array}{cccccc|cccccc} 1 & b_{12} & b_{13} & \cdots & b_{1m} & & X & 0 & 0 & \cdots & 0 \\ 0 & 1 & c_{23} & \cdots & c_{3m} & & X & X & 0 & \cdots & 0 \\ 0 & 0 & c_{33} & \cdots & c_{3m} & & X & X & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot & & \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & c_{m3} & \cdots & c_{mm} & & X & X & 0 & \cdots & 1 \end{array}\right]$$

This reduction requires

$m(m-1)$ multiplications

and

$m(m-2)$ additions.

The patterns now become apparent, with diagonalization requiring

$m(m-0)+m(m-1)+m(m-2)+ \ . \ . \ . \ +m[m-(m-1)]$ multiplications

and

$m(m-1)+m(m-2)+m(m-3)+ \ . \ . \ . \ +m[m-(m)]$ additions

or, in summation notation

$$\sum_{i-0}^{m-1} m(m-i) \qquad \text{multiplications}$$

and

$$\sum_{i=1}^{m} m(m-i) \qquad \text{additions}$$

which in turn simplify to

$$\tfrac{1}{2}m^2(m+1) \qquad \text{multiplications}$$

and

$$\tfrac{1}{2}m^2(m-1) \qquad \text{additions.}$$

We will now consider the Gaussian reduction of the upper right half of a symmetric matrix that has been reduced to the diagonal matrix A' :

$$A' = \left[\begin{array}{ccccc|ccccc}
1 & a_{12} & \cdots & a_{1(m-1)} & a_{1m} & w_{11} & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & a_{2(m-1)} & a_{2m} & w_{21} & w_{22} & \cdots & 0 & 0 \\
\cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\
\cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\
\cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\
0 & 0 & \cdots & 1 & a_{3m} & w_{(m-1)1} & w_{(m-1)2} & \cdots & w_{(m-1)(m-1)} & 0 \\
0 & 0 & \cdots & 0 & 1 & w_{m1} & w_{m2} & \cdots & w_{m(m-1)} & w_{mm}
\end{array}\right]$$

Since the upper right section of the inverse (containing zeros in A') will eventually be defined by the lower left section, it is not necessary to perform any operations on these zeros.

Thus, where

$$
B' = \left[\begin{array}{ccccc|ccccc}
1 & a_{12} & \cdots & a_{1(m-1)} & 0 & x_{11} & * & \cdots & * & * \\
0 & 0 & & a_{2(m-1)} & 0 & X_{21} & x_{22} & \cdots & * & * \\
\cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\
\cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\
\cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\
0 & 0 & \cdots & 1 & 0 & x_{(m-1)1} & x_{(m-1)2} & \cdots & x_{(m-1)(m-1)} & * \\
0 & 0 & \cdots & 0 & 1 & w_{m1} & w_{m2} & \cdots & w_{m(m-1)} & w_{mm}
\end{array}\right]
$$

The operations converting A' to B' require

$$(m-1) + (m-2) + (m-3) + \ . \ . \ . + [m-(m-1)]$$

$$= \sum_{i=1}^{m-1} i$$

Continuing this process presents the series

$$\sum_{i=1}^{m-1} i + \sum_{i=1}^{m-2} i + \sum_{i=1}^{m-3} i + \ . \ . \ . + \sum_{i=1}^{m-(m-1)} i$$

or

$$\tfrac{1}{2}(m-1)m + \tfrac{1}{2}(m-2)(m-1) + \tfrac{1}{2}(m-3)(m-2) + \ . \ . \ . + \tfrac{1}{2}[m-(m-1)][m-(m-2)]$$

or

$$\frac{1}{2} \sum_{i=1}^{m-1} (m-i)[m-(i-1)]$$

which represents the number of multiplications or additions required to find the upper diagonal and can be simplified to

$$\frac{1}{6} m(m^2-1)$$

Total multiplications required to invert a symmetric mxm matrix are, therefore, equal to

$$\frac{1}{2} m^2(m+1)+\frac{1}{6} m(m^2-1)$$

which simplifies to

$$\frac{1}{6} m(4m^2+3m-1) \qquad \text{multiplications.}$$

Total additions, likewise, are equal to

$$\frac{1}{2} m^2(m-1)+\frac{1}{6} m(m^2-1)$$

which simplifies to

$$\frac{1}{6} m(4m^2-3m-1)$$

APPENDIX E. LEAST-SQUARES VS. ARRAY ALGEBEA: A COMPARISON OF COMPUTATIONAL TIMES AS THEY VARY WITH CHANGES IN THE LENGTH OF THE MODEL EQUATION

A comparison of multiplication and addition times in programs of polynomial fitting using both the least-squares and array algebra techniques. Assuming the use of a CDC 6400 computer. Multiplication time for the CDC 6400 is 5700 nanoseconds; addition time is 1100 nanoseconds.

THE FOLLOWING TIMES ARE BASED UPON A FITTED POLYNOMIAL OF 12 TERMS,
WITH SELECTED ARRAY ALGEBRA PARAMETERS OF 2 AND 6.

NUMBER OF OBSERVATIONS	NUMBER OF POLYNOMIALS REQUIRED FOR ONE 1:50000 MAP SHEET	SECONDS OF L-S MULTIPLICATION TIME	SECONDS OF ARR-ALG MULTIPLICATION TIME	SECONDS OF L-S ADD TIME	SECONDS OF ARR-ALG ADD TIME	SECONDS SAVED
16	163315.0	178.76	74.47	32.34	10.78	125.65
25	113526.0	194.17	71.18	35.97	10.99	147.97
36	83316.0	205.21	68.39	38.50	11.00	164.32
49	64170.0	215.14	66.57	40.67	11.01	178.23
64	50592.0	221.56	64.60	42.09	10.91	188.14
81	41032.0	227.44	63.15	43.35	10.83	196.81
100	33901.0	232.02	61.84	44.33	10.74	203.76
121	28458.0	235.69	60.67	45.11	10.64	209.48
144	24252.0	239.06	59.72	45.81	10.57	214.59
169	20829.0	241.00	58.66	46.23	10.45	218.12
196	18327.0	245.95	56.51	47.22	10.48	224.18
225	16100.0	248.07	57.82	47.66	10.41	227.49
256	14148.0	248.07	56.78	47.68	10.27	228.70
289	12648.0	250.39	56.18	48.15	10.24	231.92
324	11446.0	254.08	56.38	48.88	10.28	236.30
361	10304.0	254.89	55.80	49.05	10.20	237.94
400	9328.0	255.73	55.30	49.22	10.14	239.51
441	8568.0	259.01	55.39	49.87	10.18	243.31
484	7760.0	257.52	54.50	49.59	10.04	242.56

THE FOLLOWING TIMES ARE BASED UPON A FITTED POLYNOMIAL OF 12 TERMS,
WITH SELECTED ARRAY ALGEBRA PARAMETERS OF 3 AND 4.

NUMBER OF OBSERVATIONS	NUMBER OF POLYNOMIALS REQUIRED FOR ONE 1:50000 MAP SHEET	SECONDS OF L-S MULTIPLICATION TIME	SECONDS OF ARR-ALG MULTIPLICATION TIME	SECONDS OF L-S ADD TIME	SECONDS OF ARR-ALG ADD TIME	SECONDS SAVED
16	163315.0	178.76	89.37	32.34	12.93	108.80
25	113526.0	194.17	87.36	35.97	13.49	129.29
36	83316.0	205.21	85.48	38.50	13.75	144.48
49	64170.0	215.14	84.49	40.67	13.98	157.34
64	50592.0	221.56	83.05	42.09	14.02	166.57
81	41032.0	227.44	82.10	43.35	14.08	174.61
100	33901.0	232.02	81.16	44.33	14.10	181.08
121	28458.0	235.69	80.30	45.11	14.09	186.41
144	24252.0	239.06	79.63	45.81	14.09	191.16
169	20829.0	241.00	78.72	46.23	14.02	194.49
196	18327.0	245.95	78.98	47.22	14.15	200.04
225	16100.0	248.07	78.47	47.66	14.13	203.13
256	14148.0	248.07	77.42	47.68	14.01	204.32
289	12648.0	250.39	77.22	48.15	14.02	207.30
324	11446.0	254.08	77.51	48.88	14.13	211.32
361	10304.0	254.89	77.00	49.05	14.08	212.86
400	9328.0	255.73	76.57	49.22	14.04	214.34
441	8568.0	259.01	76.92	49.87	14.14	217.82
484	7760.0	257.52	75.91	49.59	13.98	217.22

THE FOLLOWING TIMES ARE BASED UPON A FITTED POLYNOMIAL OF 16 TERMS,
WITH SELECTED ARRAY ALGEBRA PARAMETERS OF 2 AND 8.

NUMBER OF OBSERVATIONS	NUMBER OF POLYNOMIALS REQUIRED FOR ONE 1:50000 MAP SHEET	SECONDS OF L-S MULTIPLICATION TIME	SECONDS OF ARR-ALG MULTIPLICATION TIME	SECONDS OF L-S ADD TIME	SECONDS OF ARR-ALG ADD TIME	SECONDS SAVED
25	113526.0	258.91	84.13	47.97	12.99	209.76
36	83316.0	273.64	79.79	51.34	12.83	232.36
49	64170.0	286.89	76.82	54.23	12.71	251.60
64	50592.0	295.45	73.83	56.13	12.47	265.28
81	41032.0	303.31	71.58	57.81	12.28	277.26
100	33901.0	309.42	69.57	59.11	12.08	286.87
121	28458.0	314.33	67.81	60.16	11.90	294.77
144	24252.0	318.63	66.36	61.10	11.74	301.83
169	20829.0	321.43	64.83	61.66	11.55	306.70
196	18327.0	328.05	64.36	62.98	11.53	315.14
225	16100.0	330.89	63.33	63.57	11.41	319.72
256	14148.0	330.90	61.95	63.60	11.21	321.36
289	12648.0	334.02	61.29	64.23	11.13	325.83
324	11446.0	338.96	61.08	65.20	11.13	331.95
361	10304.0	340.06	60.27	65.44	11.02	334.21
400	9328.0	341.20	59.56	65.67	10.92	336.38
441	8568.0	345.60	59.50	66.54	10.94	341.70
529	7161.0	346.68	58.22	66.77	10.75	344.48
625	6106.0	349.45	57.44	67.32	10.64	348.69
676	5644.0	349.49	56.90	67.33	10.56	349.37

THE FOLLOWING TIMES ARE BASED UPON A FITTED POLYNOMIAL OF 16 TERMS,
WITH SELECTED ARRAY ALGEBRA PARAMETERS OF 4 AND 4.

NUMBER OF OBSERVATIONS	NUMBER OF POLYNOMIALS REQUIRED FOR ONE 1:50000 MAP SHEET	SECONDS OF L-S MULTIPLICATION TIME	SECONDS OF ARR-ALG MULTIPLICATION TIME	SECONDS OF L-S ADD TIME	SECONDS OF ARR-ALG ADD TIME	SECONDS SAVED
25	113526.0	258.91	116.48	47.97	17.98	172.42
36	83316.0	273.64	113.98	51.34	18.33	192.67
49	64170.0	286.89	112.66	54.23	18.64	209.83
64	50592.0	295.45	110.74	56.13	18.70	222.14
81	41032.0	303.31	109.46	57.81	18.78	232.88
100	33901.0	309.42	108.21	59.11	18.79	241.52
121	28458.0	314.33	107.06	60.16	18.78	248.64
144	24252.0	318.63	106.17	61.10	18.78	254.99
169	20829.0	321.43	104.96	61.66	18.70	259.44
196	18327.0	328.05	105.30	62.98	18.87	266.86
225	16100.0	330.89	104.62	73.57	18.84	271.00
256	14148.0	330.90	103.23	63.60	18.68	272.61
289	12648.0	334.02	102.95	64.23	18.70	276.60
324	11446.0	338.96	103.35	65.20	18.84	281.98
361	10304.0	340.06	102.67	65.44	18.77	284.06
400	9328.0	341.20	102.09	65.67	18.72	286.06
441	8568.0	345.60	102.56	66.54	18.85	290.72
529	7161.0	346.66	101.39	66.77	18.72	293.34
625	6106.0	349.45	100.94	67.32	18.70	297.14
676	5644.0	349.49	100.38	67.33	18.63	297.82

THE FOLLOWING TIMES ARE BASED UPON A FITTED POLYNOMIAL OF 18 TERMS,
WITH SELECTED ARRAY ALGEBRA PARAMETERS OF 2 AND 9.

NUMBER OF OBSERVATIONS	NUMBER OF POLYNOMIALS REQUIRED FOR ONE 1:50000 MAP SHEET	SECONDS OF L-S MULTIPL ICATION TIME	SECONDS OF ARR-ALG MULTIPL ICATION TIME	SECONDS OF L-S ADD TIME	SECONDS OF ARR-ALG ADD TIME	SECONDS SAVED
25	113526.0	291.29	90.60	53.96	13.99	240.66
36	83316.0	307.86	85.49	57.76	13.75	266.38
49	64170.0	322.77	81.94	61.02	13.55	288.29
64	50592.0	332.41	78.45	63.15	13.25	303.86
81	41032.0	341.25	75.79	65.04	13.00	317.50
100	33901.0	348.13	73.44	66.51	12.76	328.44
121	28456.0	353.66	71.38	67.68	12.52	337.44
144	24252.0	358.74	69.68	68.75	12.33	345.47
169	20829.0	361.66	67.92	69.38	12.10	351.02
196	18327.0	369.12	67.29	70.87	12.06	360.65
225	16100.0	372.33	66.09	71.53	11.90	365.86
256	14148.0	372.35	64.53	71.57	11.67	367.72
289	12648.0	375.87	63.75	72.28	11.58	372.82
324	11446.0	381.43	63.43	73.38	11.56	379.81
361	10304.0	382.69	62.51	73.64	11.43	382.39
400	9328.0	383.97	61.69	73.91	11.31	384.87
441	8568.0	388.94	61.55	74.88	11.31	390.95
529	7161.0	390.18	60.10	75.15	11.09	394.13
625	6106.0	393.33	59.19	75.77	10.97	398.95
729	5280.0	397.00	58.53	76.49	10.88	404.09

THE FOLLOWING TIMES ARE BASED UPON A FITTED POLYNOMIAL OF 18 TERMS,
WITH SELECTED ARRAY ALGEBRA PARAMETERS OF 3 AND 6.

NUMBER OF OBSERVATIONS	NUMBER OF POLYNOMIALS REQUIRED FOR ONE 1:50000 MAP SHEET	SECONDS OF L-S MULTIPL ICATION TIME	SECONDS OF ARR-ALG MULTIPL ICATION TIME	SECONDS OF L-S ADD TIME	SECONDS OF ARR-ALG ADD TIME	SECONDS SAVED
25	113526.0	291.29	106.77	53.96	16.48	221.99
36	83316.0	307.86	102.58	57.76	16.50	246.54
49	64170.0	322.77	99.86	61.02	16.52	267.41
64	50592.0	332.41	96.90	63.15	16.36	282.30
81	41032.0	341.25	94.73	65.04	16.25	295.32
100	33901.0	348.13	92.76	66.51	16.11	305.77
121	28456.0	353.66	91.01	67.68	15.97	314.37
144	24252.0	358.74	89.58	68.75	15.85	322.05
169	20829.0	361.66	87.98	69.38	15.67	327.39
196	18327.0	369.12	87.76	70.87	15.73	336.51
225	16100.0	372.33	86.73	71.53	15.62	341.50
256	14148.0	372.35	85.17	71.57	15.41	343.35
289	12648.0	375.87	84.57	72.28	15.36	348.21
324	11446.0	381.43	84.56	73.38	15.41	354.83
361	10304.0	382.69	83.70	73.64	15.30	357.32
400	9328.0	383.97	82.95	73.91	15.21	359.72
441	8568.0	388.94	83.08	74.88	15.27	365.47
529	7161.0	390.18	81.69	75.15	15.08	368.56
625	6106.0	393.33	80.93	75.77	14.99	373.18
729	5280.0	397.00	80.46	76.49	14.95	378.08

THE FOLLOWING TIMES ARE BASED UPON A FITTED POLYNOMIAL OF 20 TERMS,
WITH SELECTED ARRAY ALGEBRA PARAMETERS OF 2 AND 10.

NUMBER OF OBSERVATIONS	NUMBER OF POLYNOMIALS REQUIRED FOR ONE 1:50000 MAP SHEET	SECONDS OF L-S MULTIPLICATION TIME	SECONDS OF ARR-ALG MULTIPLICATION TIME	SECONDS OF L-S ADD TIME	SECONDS OF ARR-ALG ADD TIME	SECONDS SAVED
25	113526.0	323.67	97.07	59.96	14.99	271.57
36	83316.0	342.09	91.19	64.18	14.67	300.41
49	64170.0	358.66	87.06	67.80	14.40	324.99
64	50592.0	369.37	83.06	70.17	14.03	342.45
81	41032.0	379.20	80.00	72.27	13.72	357.75
100	33901.0	386.85	77.31	73.91	13.43	370.02
121	28458.0	393.00	74.96	75.21	13.15	380.11
144	24252.0	398.65	73.00	76.40	12.91	389.13
169	20829.0	401.91	71.01	77.10	12.65	395.35
196	18327.0	410.21	70.22	78.76	12.58	406.17
225	16100.0	413.78	68.85	79.49	12.40	412.03
256	14148.0	413.82	67.11	79.54	12.14	414.10
289	12648.0	417.74	66.20	80.33	12.02	419.84
324	11446.0	423.93	65.78	81.55	11.99	427.70
361	10304.0	425.34	64.75	81.85	11.84	430.60
400	9328.0	426.78	63.83	82.15	11.70	433.40
441	8568.0	432.31	63.61	83.23	11.69	440.24
529	7161.0	433.72	61.99	83.53	11.44	443.82
625	6106.0	437.26	60.93	84.23	11.29	449.27
729	5280.0	441.37	60.16	85.04	11.18	455.07
841	4650.0	448.77	59.99	86.48	11.18	464.09

THE FOLLOWING TIMES ARE BASED UPON A FITTED POLYNOMIAL OF 20 TERMS,
WITH SELECTED ARRAY ALGEBRA PARAMETERS OF 4 AND 5.

NUMBER OF OBSERVATIONS	NUMBER OF POLYNOMIALS REQUIRED FOR ONE 1:50000 MAP SHEET	SECONDS OF L-S MULTIPLICATION TIME	SECONDS OF ARR-ALG MULTIPLICATION TIME	SECONDS OF L-S ADD TIME	SECONDS OF ARR-ALG ADD TIME	SECONDS SAVED
25	113526.0	323.67	129.42	59.96	19.98	234.23
36	83316.0	342.09	125.38	64.18	20.16	260.73
49	64170.0	358.66	122.90	67.80	20.33	283.22
64	50592.0	369.37	119.97	70.17	20.26	299.32
81	41032.0	379.20	117.88	72.27	20.22	313.37
100	33901.0	386.85	115.95	73.91	20.14	324.67
121	28458.0	393.00	114.20	75.21	20.04	333.98
144	24252.0	398.65	112.81	76.40	19.96	342.29
169	20829.0	401.91	111.13	77.10	19.80	348.08
196	18327.0	410.21	111.16	78.76	19.92	357.89
225	16100.0	413.78	110.13	79.49	19.84	363.30
256	14148.0	413.82	108.39	79.54	19.61	365.36
289	12648.0	417.74	107.86	80.33	19.59	370.62
324	11446.0	423.93	108.05	81.55	19.69	377.74
361	10304.0	425.34	107.14	81.85	19.59	380.46
400	9328.0	426.78	106.35	82.15	19.50	383.08
441	8568.0	432.31	106.67	83.23	19.60	389.27
529	7161.0	433.72	105.16	83.53	19.41	392.69
625	6106.0	437.26	104.42	84.23	19.35	397.72
729	5280.0	441.37	104.02	85.04	19.33	403.06
841	4650.0	448.77	104.55	86.48	19.48	411.23

THE FOLLOWING TIMES ARE BASED UPON A FITTED POLYNOMIAL OF 36 TERMS, WITH SELECTED ARRAY ALGEBRA PARAMETERS OF 2 AND 18.

NUMBER OF OBSERVATIONS	NUMBER OF POLYNOMIALS REQUIRED FOR ONE 1:50000 MAP SHEET	SECONDS OF L-S MULTIPLICATION TIME	SECONDS OF ARR-ALG MULTIPLICATION TIME	SECONDS OF L-S ADD TIME	SECONDS OF ARR-ALG ADD TIME	SECONDS SAVED
49	64170.0	645.95	126.06	122.11	21.18	618.81
64	50592.0	665.31	120.01	126.39	20.27	651.42
81	41032.0	683.09	113.72	130.19	19.51	680.06
100	33901.0	696.95	108.26	133.15	18.80	703.03
121	28456.0	708.12	103.54	135.52	18.17	721.93
144	24252.0	718.41	99.59	137.67	17.62	738.88
169	20829.0	724.40	95.75	138.96	17.06	750.55
196	18327.0	739.47	93.66	141.97	16.78	770.99
225	16100.0	746.03	90.92	143.32	16.38	782.06
256	14148.0	746.26	87.81	143.44	15.89	786.00
289	12648.0	753.47	85.86	144.89	15.60	796.91
324	11446.0	764.79	84.63	147.12	15.42	811.86
361	10304.0	767.51	82.66	147.69	15.11	817.43
400	9328.0	770.30	80.90	148.27	14.83	822.83
441	8568.0	780.46	80.08	150.26	14.72	835.92
529	7161.0	783.43	77.07	150.88	14.23	843.01
625	6106.0	790.27	74.92	152.24	13.88	853.70
729	5280.0	798.17	73.23	153.79	13.61	865.12
841	4650.0	812.05	72.36	156.49	13.48	882.70
961	4060.0	811.55	70.42	156.41	13.15	884.39
1089	3564.0	808.78	68.50	155.89	12.82	883.36
1225	3162.0	808.71	66.99	155.89	12.56	885.06
1369	2891.0	827.63	67.20	159.55	12.62	907.36
1521	2576.0	821.18	65.42	158.31	12.30	901.78

THE FOLLOWING TIMES ARE BASED UPON A FITTED POLYNOMIAL OF 36 TERMS, WITH SELECTED ARRAY ALGEBRA PARAMETERS OF 4 AND 9.

NUMBER OF OBSERVATIONS	NUMBER OF POLYNOMIALS REQUIRED FOR ONE 1:50000 MAP SHEET	SECONDS OF L-S MULTIPLICATION TIME	SECONDS OF ARR-ALG MULTIPLICATION TIME	SECONDS OF L-S ADD TIME	SECONDS OF ARR-ALG ADD TIME	SECONDS SAVED
49	64170.0	645.95	163.87	122.11	27.11	577.08
64	50592.0	665.31	156.89	126.39	26.49	608.32
81	41032.0	683.09	151.57	130.19	26.00	635.72
100	33901.0	696.95	146.87	133.15	25.51	657.72
121	28458.0	708.12	142.76	135.52	25.05	675.84
144	24252.0	718.41	139.36	137.67	24.65	692.07
169	20829.0	724.40	135.84	138.96	24.20	703.32
196	18327.0	739.47	134.56	141.97	24.11	722.76
225	16100.0	746.03	132.17	143.32	23.80	733.39
256	14148.0	746.26	129.05	143.44	23.35	737.30
289	12648.0	753.47	127.48	144.89	23.15	747.73
324	11446.0	764.79	126.85	147.12	23.12	761.94
361	10304.0	767.51	125.00	147.69	22.85	767.34
400	9328.0	770.30	123.37	148.27	22.62	772.57
441	8568.0	780.46	123.09	150.26	22.62	785.00
529	7161.0	783.43	120.19	150.88	22.19	791.93
625	6106.0	790.27	118.36	152.24	21.93	802.22
729	5280.0	798.17	117.04	153.79	21.75	813.18
841	4650.0	812.05	116.86	156.49	21.77	829.91
961	4060.0	811.55	114.81	156.41	21.44	831.71
1089	3564.0	808.78	112.66	155.89	21.08	830.94
1225	3162.0	808.71	111.06	155.89	20.82	832.73
1369	2891.0	827.63	112.22	159.55	21.07	853.88
1521	2576.0	821.18	109.98	158.31	20.68	848.83

THE FOLLOWING TIMES ARE BASED UPON A FITTED POLYNOMIAL OF 36 TERMS, WITH SELECTED ARRAY ALGEBRA PARAMETERS OF 6 AND 6.

NUMBER OF OBSERVATIONS	NUMBER OF POLYNOMIALS REQUIRED FOR ONE 1:50000 MAP SHEET	SECONDS OF L-S MULTIPLICATION TIME	SECONDS OF ARR-ALG MULTIPLICATION TIME	SECONDS OF L-S ADD TIME	SECONDS OF ARR-ALG ADD TIME	SECONDS SAVED
49	64170.0	645.95	199.71	122.11	33.04	535.31
64	50592.0	665.31	193.79	126.39	32.72	565.18
81	41032.0	683.09	189.45	130.19	32.50	591.34
100	33901.0	696.95	185.51	133.15	32.22	612.37
121	28458.0	708.12	182.00	135.52	31.93	629.71
144	24252.0	718.41	179.16	137.67	31.69	645.23
169	20829.0	724.40	175.96	138.96	31.34	656.06
196	18327.0	739.47	175.50	141.97	31.45	674.48
225	16100.0	746.03	173.45	143.32	31.24	684.66
256	14148.0	746.26	170.33	143.44	30.82	688.56
289	12648.0	753.47	169.14	144.89	30.72	698.51
324	11446.0	764.79	169.11	147.12	30.82	711.97
361	10304.0	767.51	167.40	147.69	30.60	717.20
400	9328.0	770.30	165.90	148.27	30.41	722.25
441	8568.0	780.46	166.15	150.26	30.54	734.02
529	7161.0	783.43	163.36	150.88	30.15	740.79
625	6106.0	790.27	161.85	152.24	29.98	750.67
729	5280.0	798.17	160.90	153.79	29.90	761.16
841	4650.0	812.05	161.43	156.49	30.08	777.04
961	4060.0	811.55	159.27	156.41	29.75	778.94
1089	3564.0	808.78	156.88	155.89	29.36	778.43
1225	3162.0	808.71	155.19	155.89	29.09	780.32
1369	2891.0	827.63	157.32	159.55	29.54	800.32
1521	2576.0	821.18	154.63	158.31	29.08	795.79

THE FOLLOWING TIMES ARE BASED UPON A FITTED POLYNOMIAL OF 100 TERMS,
WITH SELECTED ARRAY ALGEBRA PARAMETERS OF 5 AND 20.

NUMBER OF OBSERVATIONS	NUMBER OF POLYNOMIALS REQUIRED FOR ONE 1:50000 MAP SHEET	SECONDS OF L-S MULTIPL ICATION TIME	SECONDS OF ARR-ALG MULTIPL ICATION TIME	SECONDS OF L-S ADD TIME	SECONDS OF ARR-ALG ADD TIME	SECONDS SAVED
121	28458.0	1976.96	276.64	378.36	48.53	2030.14
144	24252.0	2006.79	265.49	384.57	46.97	2078.90
169	20829.0	2024.78	254.75	388.42	45.38	2113.08
196	18327.0	2068.13	248.71	397.06	44.57	2171.92
225	16100.0	2087.96	240.96	401.13	43.41	2204.69
256	14148.0	2090.27	232.34	401.78	42.04	2217.67
289	12648.0	2112.13	226.83	406.16	41.20	2250.26
324	11446.0	2145.47	223.23	412.73	40.69	2294.28
361	10304.0	2155.05	217.71	414.70	39.80	2312.24
400	9328.0	2164.93	212.78	416.71	39.01	2329.84
441	8568.0	2195.40	210.36	422.67	38.66	2369.05
529	7161.0	2208.47	201.96	425.33	37.28	2394.56
625	6106.0	2232.71	195.90	430.12	36.29	2430.63
729	5280.0	2260.36	191.09	435.53	35.51	2469.29
841	4650.0	2305.04	188.46	444.21	35.11	2525.68
961	4060.0	2310.21	183.08	445.26	34.19	2538.20
1089	3564.0	2309.53	177.81	445.17	33.27	2543.62
1225	3162.0	2316.78	173.64	446.60	32.55	2557.19
1369	2891.0	2377.20	173.94	458.27	32.66	2628.88
1521	2576.0	2367.62	169.11	456.44	31.80	2623.16
1681	2332.0	2382.49	166.41	459.32	31.33	2644.07
1849	2142.0	2419.96	165.57	466.55	31.21	2689.74
2025	1960.0	2439.87	163.59	470.40	30.87	2715.81
2209	1833.0	2501.31	164.71	482.25	31.11	2787.74
2401	1665.0	2488.47	160.65	479.77	30.37	2777.21
2601	1548.0	2521.97	159.97	486.22	30.27	2817.96
2809	1428.0	2531.21	157.69	488.00	29.86	2831.66
3025	1320.0	2539.34	155.42	489.55	29.45	2844.03
3249	1248.0	2593.75	156.35	500.04	29.64	2907.80
3481	1178.0	2639.80	156.73	508.91	29.73	2962.24
3721	1080.0	2613.68	152.34	503.85	28.92	2936.27
3969	1015.0	2640.57	151.53	509.02	28.78	2969.29
4225	952.0	2658.92	150.18	512.54	28.53	2992.75

THE FOLLOWING TIMES ARE BASED UPON A FITTED POLYNOMIAL OF 100 TERMS, WITH SELECTED ARRAY ALGEBRA PARAMETERS OF 4 AND 25.

NUMBER OF OBSERVATIONS	NUMBER OF POLYNOMIALS REQUIRED FOR ONE 1:50000 MAP SHEET	SECONDS OF L-S MULTIPLICATION TIME	SECONDS OF ARR-ALG MULTIPLICATION TIME	SECONDS OF L-S ADD TIME	SECONDS OF ARR-ALG ADD TIME	SECONDS SAVED
121	28458.0	1976.96	257.06	378.36	45.10	2053.15
144	24252.0	2006.79	245.64	384.57	43.45	2102.27
169	20829.0	2024.78	234.73	388.[illegible]	41.62	2136.65
196	18327.0	2068.13	228.29	397.06	40.91	2196.00
225	16100.0	2087.96	220.39	401.13	39.70	2228.99
256	14148.0	2090.27	211.76	401.78	38.31	2241.98
289	12648.0	2112.13	206.06	406.16	37.43	2274.81
324	11446.0	2145.47	202.15	412.73	36.84	2319.20
361	10304.0	2155.05	196.57	414.70	35.94	2337.24
400	9328.0	2164.93	191.58	416.71	35.12	2354.93
441	8568.0	2195.40	188.89	422.67	34.72	2394.46
529	7161.0	2208.47	180.44	425.33	33.31	2420.05
625	6106.0	2232.71	174.22	430.12	32.28	2456.32
729	5280.0	2260.36	169.23	435.53	31.45	2495.22
841	4650.0	2305.04	166.25	444.21	30.98	2552.03
961	4060.0	2310.21	160.93	445.26	30.05	2564.49
1089	3564.0	2309.53	155.78	445.17	29.15	2569.78
1225	3162.0	2316.78	151.65	446.60	28.43	2583.30
1369	2891.0	2377.20	151.48	458.27	28.44	2655.56
1521	2576.0	2367.62	146.88	456.44	27.62	2649.57
1681	2332.0	2382.49	144.17	459.32	27.14	2670.50
1849	2142.0	2419.96	143.10	466.55	26.97	2716.44
2025	1960.0	2439.87	141.08	470.40	26.62	2742.57
2209	1833.0	2501.31	141.75	482.25	26.77	2815.04
2401	1665.0	2488.47	137.98	479.77	26.08	2804.17
2601	1548.0	2521.97	137.15	486.22	25.95	2845.10
2809	1428.0	2531.21	134.95	488.00	25.55	2858.71
3025	1320.0	2539.34	132.79	489.55	25.16	2870.94
3249	1248.0	2593.75	133.37	500.04	25.29	2935.13
3481	1178.0	2639.80	133.50	508.91	25.32	2989.88
3721	1080.0	2613.68	129.58	503.85	24.59	2963.36
3969	1015.0	2640.57	128.71	509.02	24.44	2996.44
4225	952.0	2658.92	127.40	512.54	24.21	3019.86

THE FOLLOWING TIMES ARE BASED UPON A FITTED POLYNOMIAL OF 100 TERMS, WITH SELECTED ARRAY ALGEBRA PARAMETERS OF 10 AND 10.

NUMBER OF OBSERVATIONS	NUMBER OF POLYNOMIALS REQUIRED FOR ONE 1:50000 MAP SHEET	SECONDS OF L-S MULTIPLICATION TIME	SECONDS OF ARR-ALG MULTIPLICATION TIME	SECONDS OF L-S ADD TIME	SECONDS OF ARR-ALG ADD TIME	SECONDS SAVED
121	28458.0	1976.96	374.72	378.36	65.74	1914.85
144	24252.0	2006.79	364.96	384.57	64.56	1961.84
169	20829.0	2024.78	355.00	388.42	63.24	1994.96
196	18327.0	2066.13	351.02	397.06	62.90	2051.28
225	16100.0	2067.96	344.15	401.13	61.99	2082.94
256	14148.0	2090.27	335.50	401.78	60.70	2095.66
289	12648.0	2112.13	330.93	406.16	60.11	2127.25
324	11446.0	2145.47	328.84	412.73	59.93	2169.42
361	10304.0	2155.05	323.64	414.70	59.17	2186.94
400	9328.0	2164.93	319.04	416.71	58.49	2204.10
441	8568.0	2195.40	317.96	422.67	58.44	2241.67
529	7161.0	2208.47	309.83	425.33	57.19	2266.77
625	6106.0	2232.71	304.56	430.12	56.42	2301.84
729	5280.0	2260.36	300.69	435.53	55.88	2339.33
841	4650.0	2305.04	299.80	444.21	55.86	2393.59
961	4060.0	2310.21	294.17	445.26	54.94	2406.37
1089	3564.0	2309.53	288.30	445.17	53.95	2412.45
1225	3162.0	2316.76	283.90	446.60	53.22	2426.25
1369	2891.0	2377.20	286.60	458.27	53.81	2495.06
1521	2576.0	2367.62	280.63	456.44	52.77	2490.66
1681	2332.0	2362.49	277.98	459.32	52.34	2511.48
1849	2142.0	2419.96	278.29	466.55	52.46	2555.77
2025	1960.0	2439.87	276.55	470.40	52.18	2581.54
2209	1833.0	2501.31	279.95	482.25	52.88	2650.74
2401	1665.0	2488.47	274.42	479.77	51.88	2641.94
2601	1546.0	2521.97	274.55	486.22	51.94	2681.70
2809	1428.0	2531.21	271.83	488.00	51.47	2695.91
3025	1320.0	2539.34	269.04	489.55	50.97	2708.88
3249	1248.0	2593.75	271.72	500.04	51.52	2770.55
3481	1176.0	2639.80	273.41	508.91	51.87	2823.43
3721	1080.0	2613.66	266.67	503.85	50.62	2800.24
3969	1015.0	2640.51	266.13	509.02	50.54	2832.92
4225	952.0	2658.92	264.60	512.54	50.28	2856.59

THE FOLLOWING TIMES ARE BASED UPON A FITTED POLYNOMIAL OF 256 TERMS,
WITH SELECTED ARRAY ALGEBRA PARAMETERS OF 4 AND 64.

NUMBER OF OBSERVATIONS	NUMBER OF POLYNOMIALS REQUIRED FOR USE 1:50000 MAP SHEET	SECONDS OF L-S MULTIPLICATION TIME	SECONDS OF ARR-ALG MULTIPLICATION TIME	SECONDS OF L-S ADD TIME	SECONDS OF ARR-ALG ADD TIME	SECONDS SAVED
289	12646.0	5559.66	398.70	1069.20	72.43	6157.94
324	11446.0	5657.17	386.83	1088.31	70.51	6288.14
361	10304.0	5694.33	372.17	1095.79	68.05	6349.91
400	9328.0	5732.93	359.01	1103.51	65.83	6411.60
441	8568.0	5824.94	350.45	1121.47	64.42	6531.54
529	7161.0	5868.44	328.13	1134.09	60.65	6633.35
625	6106.0	5983.28	311.65	1152.66	57.74	6766.55
729	5260.0	6089.59	297.75	1173.38	55.34	6909.89
841	4650.0	6242.21	287.97	1202.98	53.66	7103.57
961	4060.0	6296.42	274.72	1213.58	51.31	7183.97
1089	3564.0	6338.36	262.29	1221.78	49.09	7248.78
1225	3162.0	6403.38	252.05	1234.40	47.25	7338.47
1369	2691.0	6607.23	248.64	1273.77	46.69	7585.67
1521	2576.0	6634.60	238.32	1279.10	44.81	7630.57
1681	2332.0	6727.28	231.35	1297.01	43.56	7749.38
1849	2142.0	6880.59	227.23	1326.60	42.83	7937.13
2025	1960.0	6991.66	221.79	1348.04	41.85	8076.05
2209	1833.0	7211.77	220.70	1390.50	41.69	8339.88
2401	1665.0	7244.44	212.94	1396.80	40.25	8388.05
2601	1548.0	7398.51	209.81	1426.51	39.70	8575.51
2809	1428.0	7493.18	204.78	1444.75	38.77	8694.39
3025	1320.0	7587.74	199.93	1462.97	37.88	8812.90
3249	1248.0	7803.53	199.27	1504.57	37.78	9071.04
3481	1176.0	8000.62	198.01	1542.56	37.56	9307.61
3721	1080.0	8015.71	190.92	1545.43	36.24	9333.98
3969	1015.0	8169.24	188.39	1575.01	35.79	9520.07
4225	952.0	8303.62	185.31	1600.89	35.21	9683.99
4489	891.0	8418.91	181.68	1623.08	34.54	9825.77
4761	864.0	8737.58	184.23	1684.51	35.03	10202.83
5041	806.0	8821.05	179.66	1700.55	34.18	10307.77
5329	775.0	9080.30	180.31	1750.51	34.32	10616.19
5625	756.0	9375.92	181.93	1807.48	34.64	10966.83
5929	696.0	9411.99	176.02	1814.39	33.52	11016.82
6241	644.0	9430.35	169.68	1817.85	32.33	11046.20
6561	644.0	9910.61	176.33	1910.44	33.60	11611.11
6889	594.0	9900.24	169.18	1908.36	32.25	11607.17
7225	572.0	10148.05	169.18	1956.09	32.26	11902.70
7569	546.0	10341.02	167.62	1993.24	31.97	12134.67
7921	525.0	10576.24	167.15	2038.54	31.89	12415.74
8281	500.0	10751.92	165.02	2072.35	31.49	12627.75
8649	480.0	10974.47	164.09	2115.20	31.32	12894.26
9025	480.0	11448.78	169.68	2206.63	32.39	13453.34
9409	437.0	11342.82	160.04	2186.08	30.56	13338.30
9801	437.0	11812.72	165.29	2276.66	31.57	13892.52
10201	396.0	11681.82	155.04	2251.31	29.62	13748.57
10609	396.0	12146.59	159.95	2340.87	30.56	14296.95
11025	380.0	12382.34	172.44	2386.31	31.04	14575.18

THE FOLLOWING TIMES ARE BASED UPON A FITTED POLYNOMIAL OF 256 TERMS, WITH SELECTED ARRAY ALGEBRA PARAMETERS OF 8 AND 32.

NUMBER OF OBSERVATIONS	NUMBER OF POLYNOMIALS REQUIRED FOR ONE 1:50000 MAP SHEET	SECONDS OF L-S MULTIPL ICATION TIME	SECONDS OF ARR-ALG MULTIPL ICATION TIME	SECONDS OF L-S ADD TIME	SECONDS OF ARR-ALG ADD TIME	SECONDS SAVED
289	12648.0	5559.86	480.72	1069.20	87.31	6061.03
324	11446.0	5657.17	470.04	1088.31	85.67	6189.77
361	10304.0	5694.33	455.61	1095.79	83.30	6251.23
400	9328.0	5732.93	442.69	1103.51	81.16	6312.59
441	8566.0	5824.94	435.18	1121.47	79.98	6431.24
529	7161.0	5888.44	413.42	1134.09	76.31	6532.79
625	6106.0	5983.26	397.13	1152.66	73.57	6665.23
729	5280.0	6089.59	383.93	1173.38	71.35	6807.70
841	4650.0	6242.21	375.50	1202.98	69.97	6999.73
961	4060.0	6296.42	361.99	1213.58	67.60	7080.40
1089	3564.0	6338.38	349.04	1221.78	65.32	7145.80
1225	3162.0	6403.38	338.58	1234.40	63.47	7235.73
1369	2891.0	6607.23	337.04	1273.77	63.28	7480.67
1521	2576.0	6634.60	325.76	1279.10	61.25	7526.69
1681	2332.0	6727.28	318.79	1297.01	60.02	7645.49
1849	2142.0	6880.59	315.54	1326.60	59.46	7832.18
2025	1960.0	6991.66	310.24	1348.04	58.54	7970.91
2209	1833.0	7211.77	310.92	1390.50	58.72	8232.62
2401	1665.0	7244.44	301.93	1396.80	57.08	8282.23
2601	1548.0	7398.51	299.41	1426.51	56.65	8468.96
2809	1428.0	7493.18	293.98	1444.75	55.66	8588.29
3025	1320.0	7587.74	288.67	1462.97	54.69	8707.35
3249	1248.0	7803.53	289.36	1504.57	54.86	8963.87
3481	1178.0	8000.62	289.09	1542.56	54.84	9199.25
3721	1080.0	8015.71	280.09	1545.43	53.16	9227.89
3969	1015.0	8169.24	277.73	1575.01	52.74	9413.77
4225	952.0	8303.62	274.45	1600.89	52.15	9577.92
4489	891.0	8418.91	270.26	1623.08	51.37	9720.36
4761	864.0	8737.58	275.35	1684.51	52.36	10094.38
5041	806.0	8821.05	269.58	1700.55	51.29	10200.74
5329	775.0	9080.30	271.70	1750.51	51.71	10507.40
5625	750.0	9375.92	275.29	1807.48	52.41	10855.69
5929	696.0	9411.99	267.23	1814.38	50.90	10908.24
6241	644.0	9430.35	258.39	1817.85	49.23	10940.58
6561	644.0	9910.61	269.68	1910.44	51.40	11499.96
6889	594.0	9900.24	259.45	1908.36	49.46	11499.69
7225	572.0	10148.05	260.33	1956.09	49.64	11794.17
7569	546.0	10341.02	258.71	1993.24	49.35	12026.20
7921	525.0	10576.24	258.78	2038.54	49.37	12306.63
8281	500.0	10751.92	256.19	2072.35	48.89	12519.19
8649	480.0	10974.47	255.45	2115.20	48.76	12785.45
9025	480.0	11448.78	265.10	2206.63	50.62	13339.69
9409	437.0	11342.82	250.39	2186.08	47.82	13230.69
9801	437.0	11812.72	259.50	2276.66	49.57	13780.31
10201	396.0	11681.92	243.65	2251.31	46.55	13643.03
10609	396.0	12146.59	252.19	2340.87	48.19	14187.08
10816	396.0	12382.34	256.52	2386.31	49.02	14463.12

THE FOLLOWING TIMES ARE BASED UPON A FITTED POLYNOMIAL OF 256 TERMS.
WITH SELECTED ARRAY ALGEBRA PARAMETERS OF 16 AND 16.

NUMBER OF OBSERVATIONS	NUMBER OF POLYNOMIALS REQUIRED FOR ONE 1:50000 MAP SHEET	SECONDS OF L-S MULTIPLICATION TIME	SECONDS OF ARR-ALG MULTIPLICATION TIME	SECONDS OF L-S ADD TIME	SECONDS OF ARR-ALG ADD TIME	SECONDS SAVED
289	12038.0	5559.86	647.17	1069.20	117.55	5864.35
324	11440.0	5657.17	636.91	1088.31	116.45	5990.13
361	10304.0	5694.33	624.98	1095.79	114.26	6050.89
400	9328.0	5732.93	612.57	1103.51	112.31	6111.56
441	8568.0	5824.94	607.21	1121.47	111.60	6227.59
529	7161.0	5888.44	585.88	1134.09	108.15	6328.50
625	6106.0	5983.28	570.86	1152.66	105.76	6459.32
729	5280.0	6089.59	559.14	1173.38	103.91	6599.93
841	4650.0	6242.21	553.51	1202.98	103.13	6788.56
961	4060.0	6296.42	539.57	1213.58	100.77	6889.66
1089	3564.0	6336.38	525.67	1221.78	98.37	6936.11
1225	3162.0	6403.38	514.44	1234.40	96.52	7026.42
1369	2891.0	6607.23	517.13	1273.77	97.10	7266.76
1521	2576.0	6634.60	504.03	1279.10	94.77	7314.90
1681	2332.0	6727.26	497.14	1297.01	93.60	7433.56
1849	2142.0	6880.59	495.72	1326.60	93.44	7618.04
2025	1960.0	6991.66	490.79	1348.04	92.61	7756.29
2209	1833.0	7211.77	495.11	1390.50	93.51	8013.64
2401	1665.0	7244.44	483.76	1396.80	91.45	8066.03
2601	1548.0	7398.51	482.53	1426.51	91.29	8251.19
2809	1428.0	7493.18	476.40	1444.75	90.20	8371.33
3025	1320.0	7587.74	470.24	1462.97	89.10	8491.37
3249	1248.0	7803.53	473.74	1504.57	89.82	8744.54
3481	1178.0	8000.62	475.54	1542.56	90.21	8977.42
3721	1080.0	8015.71	462.79	1545.43	87.84	9010.51
3969	1015.0	8169.24	460.87	1575.01	87.53	9195.85
4225	952.0	8303.62	457.28	1600.89	86.89	9360.34
4489	891.0	8418.91	452.05	1623.08	85.93	9504.01
4761	864.0	8737.58	462.31	1684.51	87.92	9871.85
5041	806.0	8821.05	454.23	1700.55	86.42	9980.96
5329	775.0	9080.30	459.39	1750.51	87.44	10283.99
5625	750.0	9375.92	467.01	1807.48	88.92	10627.46
5929	696.0	9411.99	454.73	1814.38	86.61	10685.02
6241	644.0	9430.35	440.99	1817.85	84.02	10723.19
6561	644.0	9910.61	461.66	1910.44	87.99	11271.39
6889	594.0	9900.24	445.34	1908.36	84.91	11278.35
7225	572.0	10146.05	448.05	1956.09	85.45	11570.64
7569	546.0	10341.02	446.43	1993.24	85.16	11802.68
7921	525.0	10576.24	447.65	2038.54	85.42	12081.71
8281	500.0	10751.92	444.23	2072.35	84.78	12295.26
8649	480.0	10974.47	443.98	2115.20	84.76	12560.93
9025	462.0	11448.78	461.85	2206.63	88.19	13105.38
9409	437.0	11342.82	437.08	2186.09	83.48	13008.35
9801	437.0	11812.72	453.98	2276.66	86.72	13548.67
10201	396.0	11683.92	427.02	2251.31	81.59	13424.63
10609	396.0	12146.59	442.91	2340.87	84.64	13959.91
10816	396.0	12382.34	450.97	2386.31	86.19	14231.50

APPENDIX F. LEAST-SQUARES VS. ARRAY ALGEBRA: A COMPARISON OF COMPUTATIONAL TIMES AS THEY VARY WITH THE ARRAY ALGEBRA DIRECTIONAL PARAMETERS (e AND f)

A comparison of multiplication and addition times in programs of polynomial fitting using both the least-squares and array algebra techniques. Assuming the use of a CDC 6400 computer. Multiplication time for the CDC 6400 is 5700 nanoseconds; addition time is 1100 nanoseconds.

THE FOLLOWING TIMES ARE BASED UPON A FITTED POLYNOMIAL OF 4 TERMS,
WITH OPTIMAL ARRAY ALGEBRA PARAMETERS OF 2 AND 2.

NUMBER OF OBSERVATIONS	NUMBER OF POLYNOMIALS REQUIRED FOR ONE 1:50000 MAP SHEET	SECONDS OF L-S MULTIPLICATION TIME	SECONDS OF ARR-ALG MULTIPLICATION TIME	SECONDS OF L-S ADD TIME	SECONDS OF ARR-ALG ADD TIME	SECONDS SAVED
9	255663.0	52.46	43.72	9.00	5.62	12.12
16	163315.0	59.58	44.68	10.78	6.47	19.21
25	113526.0	64.71	45.30	11.99	6.99	24.41
36	83316.0	68.39	45.59	12.83	7.33	28.30
49	64170.0	71.70	46.09	13.55	7.62	31.54
64	50592.0	73.83	46.14	14.03	7.79	33.93
81	41032.0	75.79	46.31	14.45	7.94	35.98
100	33901.0	77.31	46.38	14.77	8.05	37.65
121	28456.0	78.53	46.39	15.03	8.14	39.03
144	24252.0	79.65	46.45	15.26	8.22	40.24
169	20829.0	80.28	46.30	15.40	8.25	41.13

THE FOLLOWING TIMES ARE BASED UPON A FITTED POLYNOMIAL OF 6 TERMS,
WITH OPTIMAL ARRAY ALGEBRA PARAMETERS OF 2 AND 3.

NUMBER OF OBSERVATIONS	NUMBER OF POLYNOMIALS REQUIRED FOR ONE 1:50000 MAP SHEET	SECONDS OF L-S MULTIPLICATION TIME	SECONDS OF ARR-ALG MULTIPLICATION TIME	SECONDS OF L-S ADD TIME	SECONDS OF ARR-ALG ADD TIME	SECONDS SAVED
9	255663.0	78.70	52.46	13.50	6.75	32.98
16	163315.0	89.37	52.13	16.17	7.55	45.87
25	113526.0	97.07	51.77	17.98	7.99	55.30
36	83316.0	102.59	51.29	19.25	8.25	62.30
49	64170.0	107.55	51.21	20.33	8.47	68.21
64	50592.0	110.76	50.76	21.04	8.57	72.47
81	41032.0	113.69	50.52	21.67	8.67	76.18
100	33901.0	115.97	50.24	22.16	8.73	79.16
121	28456.0	117.81	49.96	22.55	8.77	81.62
144	24252.0	119.48	49.77	22.90	8.80	83.81
169	20829.0	120.44	49.39	23.10	8.80	85.36
196	18327.0	122.91	49.73	23.60	8.91	87.87
225	16166.0	123.96	49.56	23.81	8.93	89.29

THE FOLLOWING TIMES ARE BASED UPON A FITTED POLYNOMIAL OF 8 TERMS,
WITH OPTIMAL ARRAY ALGEBRA PARAMETERS OF 2 AND 4.

NUMBER OF OBSERVATIONS	NUMBER OF POLYNOMIALS REQUIRED FOR ONE 1:50000 MAP SHEET	SECONDS OF L-S MULTIPLICATION TIME	SECONDS OF ARR-ALG MULTIPLICATION TIME	SECONDS OF L-S ADD TIME	SECONDS OF ARR-ALG ADD TIME	SECONDS SAVED
9	255663.0	104.93	61.21	18.00	7.87	53.85
16	163315.0	119.17	59.58	21.56	8.62	72.52
25	113526.0	129.44	58.24	23.98	8.99	86.18
36	83316.0	136.79	56.99	25.67	9.16	96.30
49	64170.0	143.41	56.33	27.11	9.32	104.87
64	50592.0	147.69	55.37	28.05	9.35	111.02
81	41032.0	151.60	54.73	28.90	9.39	116.38
100	33901.0	154.65	54.11	29.54	9.40	120.69
121	28458.0	157.09	53.53	30.06	9.39	124.23
144	24252.0	159.33	53.09	30.53	9.39	127.39
169	20829.0	160.62	52.48	30.81	9.35	129.60
196	18327.0	163.91	52.65	31.47	9.44	133.29
225	16100.0	165.32	52.31	31.76	9.42	135.34
256	14148.0	165.31	51.62	31.77	9.34	136.13
289	12648.0	166.85	51.48	32.08	9.35	138.10
324	11446.0	169.29	51.68	32.57	9.42	140.77

THE FOLLOWING TIMES ARE BASED UPON A FITTED POLYNOMIAL OF 9 TERMS,
WITH OPTIMAL ARRAY ALGEBRA PARAMETERS OF 3 AND 3.

NUMBER OF OBSERVATIONS	NUMBER OF POLYNOMIALS REQUIRED FOR ONE 1:50000 MAP SHEET	SECONDS OF L-S MULTIPLICATION TIME	SECONDS OF ARR-ALG MULTIPLICATION TIME	SECONDS OF L-S ADD TIME	SECONDS OF ARR-ALG ADD TIME	SECONDS SAVED
16	163315.0	134.06	78.20	24.25	11.32	68.80
25	113526.0	145.62	77.65	26.98	11.99	82.95
36	83316.0	153.90	76.93	28.87	12.37	93.46
49	64170.0	161.34	76.81	30.50	12.71	102.32
64	50592.0	166.15	76.13	31.56	12.86	108.73
81	41032.0	170.56	75.78	32.51	13.00	114.29
100	33901.0	173.99	75.36	33.24	13.09	118.77
121	28458.0	176.74	74.94	33.82	13.15	122.47
144	24252.0	179.26	74.65	34.35	13.21	125.76
169	20829.0	180.71	74.09	34.66	13.20	128.09
196	18327.0	184.42	74.59	35.41	13.37	131.87
225	16100.0	186.00	74.34	35.73	13.39	134.01
256	14148.0	185.99	73.55	35.75	13.31	134.88
289	12648.0	187.73	73.54	36.10	13.36	136.93
324	11446.0	190.48	73.99	36.64	13.48	139.65
361	10304.0	191.09	73.65	36.77	13.47	140.74

THE FOLLOWING TIMES ARE BASED UPON A FITTED POLYNOMIAL OF 10 TERMS,
WITH OPTIMAL ARRAY ALGEBRA PARAMETERS OF 2 AND 5.

NUMBER OF OBSERVATIONS	NUMBER OF POLYNOMIALS REQUIRED FOR ONE 1:50000 MAP SHEET	SECONDS OF L-S MULTIPLICATION TIME	SECONDS OF ARR-ALG MULTIPLICATION TIME	SECONDS OF L-S ADD TIME	SECONDS OF ARR-ALG ADD TIME	SECONDS SAVED
16	163315.0	148.96	67.03	26.95	9.70	99.18
25	113526.0	161.80	64.71	29.98	9.99	117.07
36	83316.0	171.00	62.69	32.08	10.08	130.31
49	64170.0	179.27	61.45	33.89	10.16	141.55
64	50592.0	184.62	59.98	35.07	10.13	149.58
81	41032.0	189.52	58.94	36.12	10.11	156.59
100	33901.0	193.33	57.97	36.93	10.07	162.22
121	28458.0	196.39	57.10	37.58	10.02	166.85
144	24252.0	199.19	56.40	38.17	9.98	170.98
169	20829.0	200.80	55.57	38.52	9.90	173.85
196	18327.0	204.93	55.58	39.34	9.96	178.73
225	16100.0	206.69	55.07	39.71	9.92	181.41
256	14148.0	206.68	54.20	39.73	9.81	182.40
289	12648.0	208.61	53.93	40.12	9.80	185.00
324	11446.0	211.68	54.03	40.72	9.85	188.52
361	10304.0	212.35	53.57	40.86	9.79	189.85
400	9328.0	213.04	53.18	41.01	9.75	191.12

THE FOLLOWING TIMES ARE BASED UPON A FITTED POLYNOMIAL OF 12 TERMS,
WITH OPTIMAL ARRAY ALGEBRA PARAMETERS OF 3 AND 4.

NUMBER OF OBSERVATIONS	NUMBER OF POLYNOMIALS REQUIRED FOR ONE 1:50000 MAP SHEET	SECONDS OF L-S MULTIPLICATION TIME	SECONDS OF ARR-ALG MULTIPLICATION TIME	SECONDS OF L-S ADD TIME	SECONDS OF ARR-ALG ADD TIME	SECONDS SAVED
16	163315.0	178.76	89.37	32.34	12.93	108.80
25	113526.0	194.17	87.36	35.97	13.49	129.29
36	83316.0	205.21	85.48	38.50	13.75	144.48
49	64170.0	215.14	84.49	40.57	13.98	157.34
64	50592.0	221.56	83.05	42.09	14.02	166.57
81	41032.0	227.44	82.10	43.35	14.08	174.61
100	33901.0	232.02	81.16	44.33	14.10	181.08
121	28458.0	235.69	80.30	45.11	14.09	186.41
144	24252.0	239.06	79.63	45.81	14.09	191.16
169	20829.0	241.00	78.72	46.23	14.02	194.49
196	18327.0	245.95	78.98	47.22	14.15	200.04
225	16100.0	246.07	78.47	47.66	14.13	203.13
256	14148.0	248.07	77.42	47.68	14.01	204.32
289	12648.0	250.39	77.22	48.15	14.02	207.30
324	11446.0	254.08	77.51	48.88	14.13	211.32
361	10304.0	254.89	77.00	49.05	14.08	212.86
400	9328.0	255.73	76.57	49.22	14.04	214.34
441	8565.0	259.01	76.92	49.87	14.14	217.82
484	7760.0	257.52	75.91	49.59	13.98	217.22

THE FOLLOWING TIMES ARE BASED UPON A FITTED POLYNOMIAL OF 14 TERMS,
WITH OPTIMAL ARRAY ALGEBRA PARAMETERS OF 2 AND 7.

NUMBER OF OBSERVATIONS	NUMBER OF POLYNOMIALS REQUIRED FOR ONE 1:50000 MAP SHEET	SECONDS OF L-S MULTIPL ICATION TIME	SECONDS OF ARR-ALG MULTIPL ICATION TIME	SECONDS OF L-S ADD TIME	SECONDS OF ARR-ALG ADD TIME	SECONDS SAVED
16	163315.0	208.56	81.92	37.73	11.86	152.51
25	113526.0	226.54	77.66	41.97	11.99	178.86
36	83316.0	239.42	74.09	44.92	11.91	198.34
49	64170.0	251.01	71.70	47.45	11.86	214.91
64	50592.0	258.50	69.22	49.11	11.69	226.71
81	41032.0	265.37	67.36	50.58	11.56	237.03
100	33901.0	270.71	65.71	51.72	11.41	245.31
121	28458.0	275.00	64.24	52.63	11.27	252.12
144	24252.0	278.94	63.04	53.45	11.15	258.20
169	20829.0	281.21	61.74	53.94	11.00	262.41
196	18327.0	287.00	61.43	55.10	11.01	269.65
225	16100.0	289.47	60.58	55.61	10.91	273.60
256	14148.0	289.48	59.36	55.64	10.74	275.02
289	12648.0	292.20	58.84	56.19	10.69	278.86
324	11446.0	296.51	58.73	57.04	10.70	284.11
361	10304.0	297.47	58.04	57.24	10.61	286.06
400	9326.0	298.45	57.43	57.44	10.53	287.93
441	8568.0	302.29	57.44	58.20	10.56	292.49
529	7161.0	303.21	56.34	58.40	10.40	294.87
576	6586.0	303.72	55.87	58.50	10.33	296.02

THE FOLLOWING TIMES ARE BASED UPON A FITTED POLYNOMIAL OF 16 TERMS,
WITH OPTIMAL ARRAY ALGEBRA PARAMETERS OF 4 AND 4.

NUMBER OF OBSERVATIONS	NUMBER OF POLYNOMIALS REQUIRED FOR ONE 1:50000 MAP SHEET	SECONDS OF L-S MULTIPL ICATION TIME	SECONDS OF ARR-ALG MULTIPL ICATION TIME	SECONDS OF L-S ADD TIME	SECONDS OF ARR-ALG ADD TIME	SECONDS SAVED
25	113526.0	258.91	116.48	47.97	17.98	172.42
36	83316.0	273.64	113.98	51.34	18.33	192.67
49	64170.0	286.89	112.66	54.23	18.64	209.83
64	50592.0	295.45	110.74	56.13	18.70	222.14
81	41032.0	303.31	109.46	57.81	18.78	232.88
100	33901.0	309.42	108.21	59.11	18.79	241.52
121	28458.0	314.33	107.06	60.16	18.78	248.64
144	24252.0	318.83	106.17	61.10	18.76	254.99
169	20829.0	321.43	104.96	61.66	18.70	259.44
196	18327.0	328.05	105.30	62.98	18.87	266.86
225	16100.0	330.89	104.62	63.57	18.84	271.00
256	14148.0	330.90	103.23	63.60	18.68	272.61
289	12648.0	334.02	102.95	64.23	18.70	276.60
324	11446.0	338.96	103.35	65.20	18.84	281.98
361	10304.0	340.06	102.67	65.44	18.77	284.06
400	9326.0	341.20	102.09	65.67	18.72	286.06
441	8568.0	345.60	102.56	66.54	18.85	290.72
529	7161.0	346.68	101.39	66.77	18.72	293.34
625	6106.0	349.49	100.94	67.32	18.70	297.14
676	5644.0	349.49	100.38	67.33	18.63	297.82

THE FOLLOWING TIMES ARE BASED UPON A FITTED POLYNOMIAL OF 16 TERMS,
WITH OPTIMAL ARRAY ALGEBRA PARAMETERS OF 3 AND 6.

NUMBER OF OBSERVATIONS	NUMBER OF POLYNOMIALS REQUIRED FOR ONE 1:50000 MAP SHEET	SECONDS OF L-S MULTIPLICATION TIME	SECONDS OF ARR-ALG MULTIPLICATION TIME	SECONDS OF L-S ADD TIME	SECONDS OF ARR-ALG ADD TIME	SECONDS SAVED
25	113526.0	291.29	100.77	53.96	16.48	221.99
36	83316.0	307.86	102.58	57.76	16.50	246.54
49	64170.0	322.77	99.86	61.02	16.52	267.41
64	50592.0	332.41	96.90	63.15	16.36	282.30
81	41032.0	341.25	94.73	65.04	16.25	295.32
100	33901.0	348.13	92.76	66.51	16.11	305.77
121	28453.0	353.66	91.01	67.68	15.97	314.37
144	24252.0	358.74	89.56	68.75	15.85	322.05
169	20829.0	361.66	87.98	69.38	15.67	327.39
196	18327.0	369.12	87.76	70.87	15.73	336.51
225	16100.0	372.33	86.73	71.53	15.62	341.50
256	14148.0	372.35	85.17	71.57	15.41	343.35
289	12648.0	375.87	84.57	72.28	15.36	348.21
324	11446.0	381.43	84.56	73.38	15.41	354.83
361	10304.0	382.69	83.70	73.64	15.30	357.32
400	9328.0	383.97	82.95	73.91	15.21	359.72
441	8568.0	388.94	83.08	74.88	15.27	365.47
529	7161.0	390.18	81.69	75.15	15.08	368.56
625	6106.0	393.33	80.93	75.77	14.99	373.18
729	5280.0	397.00	80.46	76.49	14.95	378.06

THE FOLLOWING TIMES ARE BASED UPON A FITTED POLYNOMIAL OF 20 TERMS,
WITH OPTIMAL ARRAY ALGEBRA PARAMETERS OF 4 AND 5.

NUMBER OF OBSERVATIONS	NUMBER OF POLYNOMIALS REQUIRED FOR ONE 1:50000 MAP SHEET	SECONDS OF L-S MULTIPLICATION TIME	SECONDS OF ARR-ALG MULTIPLICATION TIME	SECONDS OF L-S ADD TIME	SECONDS OF ARR-ALG ADD TIME	SECONDS SAVED
25	113526.0	323.67	129.42	59.96	19.98	234.23
36	83316.0	342.09	125.38	64.18	20.16	260.73
49	64170.0	358.66	122.90	67.80	20.33	283.22
64	50592.0	369.37	119.97	70.17	20.26	299.32
81	41032.0	379.20	117.88	72.27	20.22	313.37
100	33901.0	386.85	115.95	73.91	20.14	324.67
121	28453.0	393.00	114.20	75.21	20.04	333.98
144	24252.0	398.65	112.81	76.40	19.96	342.29
169	20829.0	401.91	111.13	77.10	19.80	348.08
196	18327.0	410.21	111.16	78.76	19.92	357.89
225	16100.0	413.78	110.13	79.49	19.84	363.30
256	14148.0	413.82	108.39	79.54	19.61	365.36
289	12648.0	417.74	107.86	80.33	19.59	370.62
324	11446.0	423.93	108.05	81.55	19.69	377.74
361	10304.0	425.34	107.14	81.85	19.59	380.46
400	9328.0	426.76	106.35	82.15	19.50	383.06
441	8568.0	432.31	106.67	83.23	19.60	389.27
529	7161.0	433.72	105.16	83.53	19.41	392.69
625	6106.0	437.26	104.42	84.23	19.35	397.72
729	5280.0	441.37	104.02	85.04	19.33	403.06
[illegible]	[illegible]	445.77	104.55	86.44	19.36	411.23

THE FOLLOWING TIMES ARE BASED UPON A FITTED POLYNOMIAL OF 21 TERMS,
WITH OPTIMAL ARRAY ALGEBRA PARAMETERS OF 3 AND 7.

NUMBER OF OBSERVATIONS	NUMBER OF POLYNOMIALS REQUIRED FOR ONE 1:50000 MAP SHEET	SECONDS OF L-S MULTIPLICATION TIME	SECONDS OF ARR-ALG MULTIPLICATION TIME	SECONDS OF L-S ADD TIME	SECONDS OF ARR-ALG ADD TIME	SECONDS SAVED
25	113526.0	339.66	116.48	62.96	17.98	268.30
36	83316.0	359.20	111.13	67.39	17.87	297.59
49	64170.0	376.60	107.54	71.19	17.79	322.46
64	50592.0	387.86	103.82	73.66	17.53	340.18
81	41032.0	398.18	101.04	75.89	17.33	355.70
100	33401.0	406.21	98.56	77.61	17.12	368.15
121	28458.0	412.68	96.36	78.98	16.91	378.39
144	24252.0	418.61	94.56	80.22	16.73	387.55
169	20829.0	422.04	92.61	80.96	16.50	393.89
196	18327.0	430.76	92.15	82.70	16.51	404.80
225	16100.0	434.51	90.86	83.48	16.37	410.76
256	14146.0	434.56	89.04	83.53	16.11	412.94
289	12648.0	438.66	88.25	84.36	16.03	418.74
324	11446.0	445.19	88.09	85.64	16.05	426.68
361	10304.0	446.67	87.05	85.95	15.92	429.66
400	9326.0	448.19	86.15	86.27	15.79	432.52
441	8568.0	454.01	86.16	87.41	15.84	439.42
529	7161.0	455.51	84.51	87.73	15.60	443.13
625	6106.0	459.24	83.54	88.47	15.48	448.68
729	5280.0	463.57	82.90	89.32	15.41	454.58
841	4650.0	471.36	83.03	90.84	15.47	463.70

THE FOLLOWING TIMES ARE BASED UPON A FITTED POLYNOMIAL OF 22 TERMS,
WITH OPTIMAL ARRAY ALGEBRA PARAMETERS OF 2 AND 11.

NUMBER OF OBSERVATIONS	NUMBER OF POLYNOMIALS REQUIRED FOR ONE 1:50000 MAP SHEET	SECONDS OF L-S MULTIPLICATION TIME	SECONDS OF ARR-ALG MULTIPLICATION TIME	SECONDS OF L-S ADD TIME	SECONDS OF ARR-ALG ADD TIME	SECONDS SAVED
25	113526.0	356.05	103.55	65.98	15.99	302.49
36	83316.0	376.31	96.89	70.60	15.58	334.45
49	64170.0	394.55	92.19	74.59	15.25	361.70
64	50592.0	406.34	87.68	77.19	14.81	381.05
81	41032.0	417.16	84.21	79.51	14.45	398.01
100	33401.0	425.58	81.18	81.31	14.10	411.61
121	28458.0	432.35	78.53	82.74	13.76	422.79
144	24252.0	438.58	76.33	84.05	13.50	432.80
169	20829.0	442.17	74.10	84.82	13.20	439.69
196	18327.0	451.31	73.15	86.65	13.11	451.71
225	16100.0	455.25	71.60	87.46	12.90	458.21
256	14146.0	455.30	69.70	87.52	12.61	460.51
289	12648.0	459.63	68.66	88.39	12.47	466.89
324	11446.0	466.45	68.14	89.73	12.42	475.62
361	10304.0	468.01	66.98	90.06	12.25	478.85
400	9326.0	469.61	65.96	90.39	12.09	481.96
441	8568.0	475.72	65.57	91.59	12.07	489.67
529	7161.0	477.30	63.87	91.92	11.79	493.56
625	6106.0	481.23	62.88	92.70	11.61	499.44
729	5280.0	485.76	61.73	93.60	11.48	506.11
841	4650.0	493.97	61.53	95.19	11.46	516.17
900	4320.0	491.58	60.81	94.70	11.31	514.16

THE FOLLOWING TIMES ARE BASED UPON A FITTED POLYNOMIAL OF 24 TERMS, WITH OPTIMAL ARRAY ALGEBRA PARAMETERS OF 4 AND 6.

NUMBER OF OBSERVATIONS	NUMBER OF POLYNOMIALS REQUIRED FOR ONE 1:50000 MAP SHEET	SECONDS OF L-S MULTIPLICATION TIME	SECONDS OF ARR-ALG MULTIPLICATION TIME	SECONDS OF L-S ADD TIME	SECONDS OF ARR-ALG ADD TIME	SECONDS SAVED
25	113526.0	368.44	142.37	71.96	21.98	296.06
36	83316.0	410.55	136.78	77.03	22.00	328.80
49	64176.0	430.44	133.14	81.37	22.02	356.65
64	50592.0	443.32	129.20	84.21	21.82	376.52
81	41032.0	455.13	126.30	86.74	21.67	393.90
100	33901.0	464.32	123.68	88.71	21.48	407.87
121	28458.0	471.72	121.34	90.28	21.29	419.37
144	24252.0	476.52	119.44	91.70	21.13	429.65
169	20829.0	482.45	117.31	92.55	20.90	436.79
196	18327.0	492.43	117.01	94.54	20.97	449.00
225	16100.0	496.74	115.64	95.43	20.83	455.70
256	14148.0	496.81	113.55	95.49	20.54	458.20
289	12648.0	501.54	112.76	96.45	20.48	464.74
324	11446.0	509.00	112.75	97.92	20.55	473.61
361	10304.0	510.72	111.80	98.28	20.40	476.99
400	9328.0	512.48	110.60	98.64	20.28	480.24
441	8568.0	519.15	110.77	99.95	20.36	487.97
529	7161.0	520.92	108.91	100.32	20.10	492.22
625	6108.0	525.24	107.91	101.18	19.99	498.52
729	5283.0	530.25	107.28	102.17	19.94	505.21
841	4650.0	539.23	107.63	103.92	20.05	515.47
961	4060.0	538.60	106.19	103.81	19.83	516.38

THE FOLLOWING TIMES ARE BASED UPON A FITTED POLYNOMIAL OF 25 TERMS, WITH OPTIMAL ARRAY ALGEBRA PARAMETERS OF 5 AND 5.

NUMBER OF OBSERVATIONS	NUMBER OF POLYNOMIALS REQUIRED FOR ONE 1:50000 MAP SHEET	SECONDS OF L-S MULTIPLICATION TIME	SECONDS OF ARR-ALG MULTIPLICATION TIME	SECONDS OF L-S ADD TIME	SECONDS OF ARR-ALG ADD TIME	SECONDS SAVED
36	83316.0	427.67	156.72	80.24	25.20	325.98
49	64176.0	448.39	153.63	84.77	25.41	354.12
64	50592.0	461.81	149.96	87.73	25.32	374.25
81	41032.0	474.11	147.35	90.36	25.28	391.85
100	33901.0	483.69	144.93	92.41	25.17	406.00
121	28458.0	491.40	142.75	94.05	25.04	417.66
144	24252.0	498.49	141.00	95.53	24.94	428.07
169	20829.0	502.59	138.91	96.41	24.75	435.35
196	18327.0	513.00	138.94	98.49	24.90	447.65
225	16100.0	517.49	137.66	99.42	24.79	454.45
256	14148.0	517.57	135.49	99.48	24.51	457.05
289	12648.0	522.50	134.82	100.48	24.49	463.67
324	11446.0	530.28	135.06	102.01	24.62	472.62
361	10304.0	532.08	133.92	102.39	24.48	476.07
400	9328.0	533.92	132.93	102.77	24.37	479.39
441	8568.0	540.88	133.33	104.13	24.51	487.18
529	7161.0	542.74	131.49	104.53	24.28	491.57
625	6108.0	547.26	130.52	105.43	24.19	497.98
729	5283.0	552.51	130.02	106.46	24.16	504.78
841	4650.0	561.88	130.08	[illegible]	24.35	515.14
961	4060.0	561.25	129.14	108.17	24.12	516.17
1024	3808.0	561.27	128.51	108.18	24.02	515.92

THE FOLLOWING TIMES ARE BASED UPON A FITTED POLYNOMIAL OF 36 TERMS,
WITH OPTIMAL ARRAY ALGEBRA PARAMETERS OF 6 AND 6.

NUMBER OF OBSERVATIONS	NUMBER OF POLYNOMIALS REQUIRED FOR ONE 1:50000 MAP SHEET	SECONDS OF L-S MULTIPLICATION TIME	SECONDS OF ARR-ALG MULTIPLICATION TIME	SECONDS OF L-S ADD TIME	SECONDS OF ARR-ALG ADD TIME	SECONDS SAVED
49	64170.0	645.95	199.71	122.11	33.04	535.31
64	50592.0	665.31	193.79	126.39	32.72	565.18
81	41032.0	683.09	189.45	130.19	32.50	591.34
100	33901.0	696.95	185.51	133.15	32.22	612.37
121	28458.0	708.12	182.00	135.52	31.93	629.71
144	24252.0	718.41	179.16	137.67	31.69	645.23
169	20829.0	724.40	175.96	138.96	31.34	656.06
196	18327.0	739.47	175.50	141.97	31.45	674.48
225	16100.0	746.03	173.45	143.32	31.24	684.66
256	14148.0	746.26	170.33	143.44	30.82	688.56
289	12648.0	753.47	169.14	144.89	30.72	698.51
324	11446.0	764.79	169.11	147.12	30.82	711.97
361	10304.0	767.51	167.40	147.69	30.60	717.20
400	9328.0	770.30	165.90	148.27	30.41	722.25
441	8566.0	780.46	166.15	150.26	30.54	734.02
529	7161.0	783.43	163.36	150.88	30.15	740.79
625	6108.0	790.27	161.85	152.24	29.98	750.67
729	5280.0	798.17	160.90	153.79	29.90	761.16
841	4650.0	812.05	161.43	156.49	30.08	777.04
961	4060.0	811.55	159.27	156.41	29.75	778.94
1089	3564.0	808.78	156.88	155.89	29.36	778.43
1225	3162.0	808.71	155.19	155.89	29.09	780.32
1369	2891.0	827.63	157.32	159.55	29.54	800.32
1521	2576.0	821.18	154.63	158.31	29.08	795.79

THE FOLLOWING TIMES ARE BASED UPON A FITTED POLYNOMIAL OF 49 TERMS,
WITH CRITICAL ARRAY ALGEBRA PARAMETERS OF 7 AND 7.

NUMBER OF [illegible]	NUMBER OF [illegible] REQUIRED FOR [illegible] THE ARRAY	SECONDS OF L-S MULTIPLICATION TIME	SECONDS OF ARR-ALG MULTIPLICATION TIME	SECONDS OF L-S ADD TIME	SECONDS OF ARR-ALG ADD TIME	SECONDS SAVED
64	50592.0	906.12	242.24	172.13	40.90	795.11
81	41032.0	930.41	235.76	177.33	40.44	831.54
100	33981.0	949.38	229.96	181.38	39.94	860.86
121	28458.0	964.70	224.83	184.63	39.44	885.05
144	24252.0	978.63	220.63	187.58	39.03	906.74
169	20825.0	987.11	216.09	189.36	38.49	921.89
196	18327.0	1007.78	214.99	193.48	38.53	947.74
225	16100.0	1016.87	212.00	195.35	38.18	962.04
256	14148.0	1017.34	207.75	195.55	37.59	967.55
289	12046.0	1027.34	205.91	197.56	37.40	981.60
324	11446.0	1042.94	205.52	200.63	37.46	1000.58
361	10304.0	1046.64	203.11	201.44	37.13	1008.04
400	9328.0	1050.85	200.99	202.27	36.85	1015.27
441	8568.0	1064.90	201.03	205.02	36.95	1031.94
529	7161.0	1069.42	197.16	205.96	36.39	1041.83
625	6106.0	1079.25	194.92	207.91	36.11	1056.13
729	5280.0	1090.58	193.41	210.13	35.94	1071.36
841	4650.0	1110.08	193.71	213.93	36.09	1094.20
961	4060.0	1110.05	190.84	213.95	35.64	1097.51
1089	3564.0	1106.98	187.72	213.37	35.13	1097.50
1225	3162.0	1107.63	185.48	213.51	34.77	1100.89
1369	2891.0	1134.16	187.81	218.64	35.26	1129.72
1521	2576.0	1126.21	184.41	217.11	34.68	1124.24
1681	2332.0	1130.06	183.14	217.87	34.46	1130.33
1849	2142.0	1144.85	183.77	220.72	34.64	1147.16
2025	1980.0	1150.85	183.02	221.87	34.53	1155.17

THE FOLLOWING TIMES ARE BASED UPON A FITTED POLYNOMIAL OF 56 TERMS, WITH OPTIMAL ARR-[illegible]GERER PARAMETERS OF [illegible] AND [illegible].

NUMBER OF OBSERVATIONS	NUMBER OF POLYNOMIALS REQUIRED FOR ONE 1:50000 MAP SHEET	SECONDS OF L-S MULTIPLICATION TIME	SECONDS OF ARR-ALG MULTIPLICATION TIME	SECONDS OF L-S ADD TIME	SECONDS OF ARR-ALG ADD TIME	SECONDS SAVED
81	41032.0	1216.31	286.28	231.82	49.11	1112.74
100	33961.0	1241.24	276.27	237.14	48.33	1151.78
121	28458.0	1261.43	271.22	241.41	47.58	1184.03
144	24292.0	1280.06	265.42	245.31	46.95	1212.99
169	[illegible]	1291.09	259.31	247.67	46.19	1233.27
196	18327.0	1318.30	257.41	253.10	46.13	1267.87
225	16100.0	1330.42	253.30	255.59	45.62	1287.09
256	14148.0	1331.28	247.75	255.89	44.82	1294.61
289	12646.0	1344.62	245.13	258.57	44.52	1313.54
324	11446.0	1365.28	244.28	262.64	44.52	1339.11
361	10304.0	1370.68	241.05	263.76	44.07	1349.32
400	9328.0	1376.23	236.21	264.90	43.67	1359.24
441	8568.0	1394.92	237.95	268.55	43.73	1381.80
529	7161.0	1401.55	232.84	269.93	42.98	1395.66
625	6106.0	1415.16	229.72	272.62	42.56	1415.52
729	5280.0	1430.83	227.54	275.70	42.29	1436.70
841	4650.0	1457.22	227.54	280.82	42.40	1468.11
961	4060.0	1458.16	223.85	281.04	41.81	1473.55
1089	3564.0	1455.20	219.91	280.49	41.15	1474.64
1225	3162.0	1457.17	217.02	280.89	40.68	1480.35
1369	2891.0	1493.00	219.52	287.82	41.22	1520.08
1521	2576.0	1483.87	215.34	286.07	40.49	1514.11
1681	2332.0	1490.23	213.66	287.30	40.23	1523.65
1849	2142.0	1510.91	214.23	291.29	40.38	1547.59
2025	1960.0	1520.19	213.19	293.08	40.23	1559.85
2209	1833.0	1555.88	216.10	299.97	40.81	1598.94
2401	1665.0	1543.88	212.09	297.65	40.09	1589.35
2601	1548.0	1561.38	212.43	301.02	40.19	1609.78
2704	1470.0	1546.23	209.17	298.10	39.59	1595.57

THE FOLLOWING TIMES ARE BASED UPON A FITTED POLYNOMIAL OF 81 TERMS,
WITH OPTIMAL ARRAY ALGEBRA PARAMETERS OF 9 AND 9.

NUMBER OF OBSERVATIONS	NUMBER OF POLYNOMIALS REQUIRED FOR ONE 1:50000 MAP SHEET	SECONDS OF L-S MULTIPL ICATION TIME	SECONDS OF ARR-ALG MULTIPL ICATION TIME	SECONDS OF L-S ADD TIME	SECONDS OF ARR-ALG ADD TIME	SECONDS SAVED
100	33901.0	1572.88	330.44	300.50	57.39	1485.54
121	28458.0	1598.68	321.19	305.96	56.35	1527.10
144	24252.0	1622.54	313.53	310.94	55.46	1564.48
169	20629.0	1636.79	305.61	313.99	54.44	1590.73
196	18327.0	1671.55	302.75	320.92	54.25	1635.46
225	16180.0	1687.22	297.35	324.14	53.56	1660.45
256	14148.0	1688.68	290.33	324.59	52.53	1670.42
289	12648.0	1705.96	286.80	328.06	52.09	1695.12
324	11446.0	1732.50	285.39	333.28	52.01	1728.39
361	10304.0	1739.78	281.23	334.79	51.42	1741.92
400	9328.0	1747.26	277.56	336.31	50.89	1755.13
441	8568.0	1771.41	276.93	341.04	50.90	1784.62
529	7161.0	1780.83	270.40	342.97	49.91	1803.50
625	6106.0	1799.21	266.27	346.60	49.33	1830.21
729	5280.0	1820.24	263.30	350.73	48.93	1858.73
841	4650.0	1854.96	262.90	357.47	48.99	1900.54
961	4060.0	1857.57	258.29	358.02	48.24	1909.06
1089	3564.0	1855.33	253.43	357.62	47.43	1912.09
1225	3162.0	1859.41	249.83	358.43	46.84	1921.18
1369	2891.0	1906.46	252.45	367.52	47.40	1974.13
1521	2576.0	1896.70	247.41	365.65	46.52	1968.42
1681	2332.0	1906.63	245.28	367.58	46.18	1982.75
1849	2142.0	1934.78	245.74	373.01	46.32	2015.73
2025	1960.0	1948.59	244.37	375.68	46.11	2033.79
2209	1833.0	1995.93	247.53	384.81	46.75	2086.46
2401	1665.0	1963.00	242.79	382.31	45.90	2076.63
2601	1548.0	2007.51	243.04	387.03	45.98	2105.52
2809	1428.0	2012.26	240.76	387.94	45.59	2113.85
3025	1320.0	2016.00	236.40	388.65	45.17	2121.08
3249	1248.0	2057.13	240.90	396.58	45.67	2167.13
3364	1210.0	2080.16	242.46	401.02	45.98	2192.74

THE FOLLOWING TIMES ARE BASED UPON A FITTED POLYNOMIAL OF 100 TERMS.
WITH OPTIMAL ARRAY ALGEBRA PARAMETERS OF 10 AND 10.

NUMBER OF OBSERVATIONS	NUMBER OF POLYNOMIALS REQUIRED FOR ONE 1:50000 MAP SHEET	SECONDS OF L-S MULTIPLICATION TIME	SECONDS OF ARR-ALG MULTIPLICATION TIME	SECONDS OF L-S ADD TIME	SECONDS OF ARR-ALG ADD TIME	SECONDS SAVED
121	24458.0	1976.96	374.72	378.36	65.74	1914.86
144	24252.0	2006.79	364.96	384.57	64.56	1961.84
169	20829.0	2024.78	355.00	388.42	63.24	1994.96
196	18327.0	2068.13	351.02	397.06	62.90	2051.28
225	16100.0	2087.96	344.15	401.13	61.99	2082.94
256	14146.0	2090.27	335.50	401.78	60.70	2095.86
289	12648.0	2112.13	330.93	406.16	60.11	2127.25
324	11446.0	2145.47	328.84	412.73	59.93	2169.42
361	10304.0	2155.05	323.64	414.70	59.17	2186.94
400	9328.0	2164.93	319.04	416.71	58.49	2204.10
441	8568.0	2195.40	317.96	422.67	58.44	2241.67
529	7161.0	2208.47	309.83	425.33	57.19	2266.77
625	6106.0	2232.71	304.56	430.12	56.42	2301.84
729	5280.0	2260.36	300.69	435.53	55.88	2339.33
841	4650.0	2305.04	299.80	444.21	55.86	2393.59
961	4060.0	2310.21	294.17	445.26	54.94	2406.37
1089	3564.0	2309.53	288.30	445.17	53.95	2412.45
1225	3162.0	2316.78	283.90	446.60	53.22	2426.25
1369	2891.0	2377.20	286.60	458.27	53.81	2495.06
1521	2576.0	2367.62	280.63	456.44	52.77	2490.66
1681	2332.0	2382.49	277.98	459.32	52.34	2511.48
1849	2142.0	2419.96	278.29	466.55	52.46	2555.77
2025	1960.0	2439.87	276.55	470.40	52.18	2581.54
2209	1833.0	2501.31	279.95	482.25	52.88	2650.74
2401	1665.0	2488.47	274.42	479.77	51.88	2641.94
2601	1548.0	2521.97	274.55	486.22	51.94	2681.70
2809	1426.0	2531.21	271.83	488.00	51.47	2695.91
3025	1320.0	2539.34	269.04	489.55	50.97	2708.88
3249	1248.0	2593.75	271.72	500.04	51.52	2770.55
3481	1178.0	2639.80	273.41	508.91	51.87	2823.43
3721	1080.0	2613.68	266.67	503.85	50.62	2800.24
3969	1015.0	2640.57	266.13	509.02	50.54	2832.92
4225	952.0	2658.92	264.60	512.54	50.28	2856.59

THE FOLLOWING TIMES ARE BASED UPON A FITTED POLYNOMIAL OF 121 TERMS, WITH OPTIMAL ARRAY ALGEBRA PARAMETERS OF 11 AND 11.

NUMBER OF OBSERVATIONS	NUMBER OF POLYNOMIALS REQUIRED FOR ONE 1:50000 MAP SHEET	SECONDS OF L-S MULTIPLICATION TIME	SECONDS OF ARR-ALG MULTIPLICATION TIME	SECONDS OF L-S ADD TIME	SECONDS OF ARR-ALG ADD TIME	SECONDS SAVED
144	21252.0	2433.48	419.70	466.35	74.25	2405.88
169	20829.0	2455.80	407.48	471.11	72.59	2446.84
196	16327.0	2508.84	402.21	481.67	72.07	2516.23
225	16100.0	2533.46	393.71	486.72	70.91	2555.54
256	14148.0	2536.92	383.24	487.64	69.34	2571.99
289	12648.0	2564.09	377.51	493.08	68.57	2611.10
324	11446.0	2605.20	374.65	501.17	68.28	2663.44
361	10304.0	2617.60	368.28	503.71	67.33	2685.69
400	9326.0	2630.39	362.64	506.30	66.48	2707.57
441	8568.0	2668.15	361.04	513.68	66.36	2754.45
529	7161.0	2665.88	351.14	517.28	64.82	2787.19
625	6106.0	2717.29	344.59	523.47	63.84	2832.33
729	5280.0	2753.02	339.70	530.46	63.13	2880.66
841	4650.0	2809.52	338.24	541.43	63.02	2949.68
961	4080.0	2818.38	331.48	543.21	61.91	2968.20
1089	3564.0	2820.33	324.51	543.63	60.73	2978.73
1225	3162.0	2832.06	319.24	545.93	59.85	2998.91
1369	2891.0	2908.30	321.97	560.66	60.46	3086.53
1521	2576.0	2900.01	315.00	559.08	59.23	3084.86
1681	2332.0	2921.47	311.78	563.23	58.70	3114.22
1849	2142.0	2970.47	311.90	572.69	58.79	3172.46
2025	1960.0	2998.38	309.74	578.08	58.45	3208.28
2209	1833.0	3076.72	313.35	593.19	59.18	3297.38
2401	1665.0	3065.35	306.98	590.99	58.03	3291.33
2601	1548.0	3110.24	306.96	599.64	58.08	3344.84
2809	1428.0	3125.94	303.77	602.66	57.51	3367.32
3025	1320.0	3140.46	300.50	605.45	56.94	3388.48
3249	1248.0	3211.17	303.36	619.08	57.52	3469.37
3481	1178.0	3271.92	305.11	630.78	57.88	3539.71
3721	1080.0	3245.54	297.48	625.67	56.47	3517.26
3969	1015.0	3283.46	296.76	632.96	56.36	3563.30
4225	952.0	3311.23	294.95	638.30	56.04	3598.54
4489	891.0	3328.85	292.03	641.67	55.51	3622.97
4761	864.0	3441.48	299.11	663.37	56.89	3748.86
5041	806.0	3447.63	294.30	663.41	55.99	3754.94

THE FOLLOWING TIMES ARE BASED UPON A FITTED POLYNOMIAL OF 144 TERMS,
WITH OPTIMAL ARRAY ALGEBRA PARAMETERS OF 12 AND 12.

NUMBER OF OBSERVATIONS	NUMBER OF POLYTOPES REQUIRED FOR USE 1:50000 MAP SHEET	SECONDS OF L-S MULTIPLICATION TIME	SECONDS OF ARR-ALG MULTIPLICATION TIME	SECONDS OF L-S ADD TIME	SECONDS OF ARR-ALG ADD TIME	SECONDS SAVED
169	20825.5	2930.74	463.05	562.22	82.49	2947.42
196	18327.0	2994.62	456.32	574.94	81.77	3031.47
225	18100.0	3024.74	446.03	581.10	80.31	3079.47
256	14148.0	3029.74	433.57	582.37	78.44	3100.11
289	12648.0	3063.01	426.53	589.02	77.47	3148.03
324	11448.0	3112.92	422.80	598.84	77.06	3211.90
361	10304.0	3128.72	415.15	602.06	75.90	3239.73
400	9328.0	3145.06	408.37	605.37	74.87	3267.18
441	8568.0	3191.16	406.17	614.38	74.65	3324.71
529	7161.0	3214.74	394.34	619.14	72.79	3366.75
625	6106.0	3254.85	386.37	627.03	71.58	3423.93
729	5280.0	3300.31	380.33	635.92	70.68	3485.21
841	4650.0	3370.72	378.22	649.58	70.47	3571.61
961	4060.0	3384.66	370.23	652.35	69.14	3597.64
1089	3564.0	3390.61	362.06	653.56	67.75	3614.36
1225	3162.0	3408.42	355.83	657.04	66.71	3642.92
1369	2891.0	3503.23	358.56	675.35	67.33	3752.69
1521	2576.0	3497.67	350.51	674.30	65.91	3755.55
1681	2332.0	3527.74	346.67	680.12	65.27	3795.92
1849	2142.0	3590.81	346.56	692.30	65.32	3871.21
2025	1960.0	3629.01	343.94	699.67	64.90	3919.84
2209	1833.0	3727.45	347.74	718.66	65.08	4032.69
2401	1665.0	3719.35	340.47	717.09	64.36	4031.61
2601	1548.0	3778.44	340.27	728.48	64.38	4102.27
2809	1428.0	3803.01	336.57	733.21	63.72	4135.93
3025	1320.0	3826.41	332.79	737.71	63.05	4168.28
3249	1248.0	3916.90	335.81	755.15	63.67	4272.57
3481	1178.0	3995.73	337.61	770.34	64.05	4364.46
3721	1080.0	3971.17	329.04	765.57	62.46	4345.25
3969	1015.0	4023.33	328.12	775.61	62.32	4408.50
4225	952.0	4063.62	326.00	783.35	61.94	4459.04
4489	891.0	4092.07	322.67	788.81	61.34	4496.87
4761	864.0	4233.81	330.39	816.13	62.83	4656.72
5041	806.0	4242.16	324.98	817.70	61.83	4673.05
5329	775.0	4348.27	329.03	838.14	62.63	4794.76
5625	750.0	4473.74	334.83	852.31	63.75	4937.46
5929	690.0	4452.11	326.35	858.10	62.16	4921.69
6084	695.0	4568.20	334.30	880.47	63.69	5050.69

THE FOLLOWING TIMES ARE BASED UPON A FITTED POLYNOMIAL OF 169 TERMS, WITH OPTIMAL ARRAY ALGEBRA PARAMETERS OF 13 AND 13.

NUMBER OF OBSERVATIONS	NUMBER OF POLYNOMIALS REQUIRED FOR ONE 1:50000 MAP SHEET	SECONDS OF L-S MULTIPLICATION TIME	SECONDS OF ARR-ALG MULTIPLICATION TIME	SECONDS OF L-S ADD TIME	SECONDS OF ARR-ALG ADD TIME	SECONDS SAVED
196	18327.0	3526.64	513.37	677.09	91.99	3598.37
225	16100.0	3563.03	501.10	684.52	90.26	3656.20
256	14148.0	3570.03	486.47	686.23	88.01	3681.76
289	12648.0	3610.26	478.01	694.26	86.82	3739.69
324	11448.0	3670.10	473.30	706.03	86.26	3816.56
361	10304.0	3689.98	464.26	710.07	84.88	3850.91
400	9328.0	3710.56	456.23	714.22	83.64	3884.90
441	8568.0	3766.14	453.35	725.08	83.32	3954.55
529	7161.0	3797.00	439.40	731.28	81.11	4007.76
625	6106.0	3847.54	429.88	741.21	79.64	4079.23
729	5280.0	3904.65	422.60	752.36	78.53	4155.89
841	4650.0	3991.33	419.73	769.19	78.21	4262.58
961	4060.0	4012.03	410.41	773.27	76.65	4298.24
1089	3564.0	4023.63	400.95	775.58	75.03	4323.23
1225	3162.0	4049.45	393.69	780.61	73.80	4362.56
1369	2891.0	4165.92	396.38	803.11	74.43	4498.22
1521	2576.0	4164.88	387.17	802.94	72.80	4507.84
1681	2332.0	4205.95	382.65	810.88	72.04	4562.14
1849	2142.0	4286.04	382.28	826.34	72.06	4658.05
2025	1960.0	4337.23	379.14	836.22	71.54	4722.77
2209	1833.0	4459.41	383.11	859.79	72.36	4863.74
2401	1665.0	4456.85	374.90	859.29	70.87	4870.37
2601	1548.0	4533.41	374.49	874.05	70.85	4962.12
2809	1428.0	4569.76	370.23	881.05	70.10	5010.49
3025	1320.0	4605.03	365.91	887.84	69.33	5057.63
3249	1248.0	4719.33	369.08	909.86	69.97	5190.14
3481	1178.0	4820.27	370.90	929.31	70.36	5308.32
3721	1080.0	4800.07	361.35	925.39	68.59	5295.52
3969	1015.0	4870.24	360.21	938.90	68.41	5380.51
4225	952.0	4926.77	357.76	949.77	67.98	5450.81
4489	891.0	4969.69	353.99	958.02	67.29	5506.42
4761	864.0	5145.88	362.35	991.97	68.91	5706.58
5041	806.0	5165.78	356.31	995.77	67.79	5737.44
5329	775.0	5300.72	360.65	1021.76	68.64	5893.19
5625	750.0	5458.66	366.91	1052.19	69.86	6074.08
5929	696.0	5444.26	357.53	1049.37	68.10	6068.00
6241	644.0	5417.16	346.96	1044.09	66.11	6048.18
6561	644.0	5693.97	363.47	1097.45	69.28	6358.67
6889	594.0	5645.90	350.85	1088.12	66.89	6316.28
7056	594.0	5782.32	358.77	1114.42	68.41	6469.55

THE FOLLOWING TIMES ARE BASED UPON A FITTED POLYNOMIAL OF 196 TERMS, WITH OPTIMAL ARRAY ALGEBRA PARAMETERS OF 14 AND 14.

NUMBER OF OBSERVATIONS	NUMBER OF POLYNOMIALS REQUIRED FOR ONE 1:50000 MAP SHEET	SECONDS OF L-S MULTIPLICATION TIME	SECONDS OF ARR-ALG MULTIPLICATION TIME	SECONDS OF L-S ADD TIME	SECONDS OF ARR-ALG ADD TIME	SECONDS SAVED
225	16100.0	4149.81	558.92	797.26	100.67	4287.48
256	14146.0	4159.32	541.96	799.51	98.05	4318.81
289	12648.0	4207.48	531.95	809.12	96.62	4388.03
324	11446.0	4276.46	526.15	823.07	95.90	4479.48
361	10304.0	4303.19	515.60	828.08	94.26	4521.41
400	9326.0	4328.82	506.22	833.23	92.81	4563.02
441	8566.0	4395.14	502.59	846.18	92.37	4646.37
529	7161.0	4434.91	486.35	854.14	89.78	4712.92
625	6106.0	4497.87	475.13	866.50	88.02	4801.21
729	5280.0	4568.81	466.49	880.34	86.69	4895.98
841	4650.0	4674.40	462.79	900.83	86.23	5026.22
961	4060.0	4703.84	452.03	906.61	84.42	5074.01
1089	3564.0	4723.09	441.18	910.41	82.56	5109.76
1225	3162.0	4759.18	432.81	917.43	81.14	5162.66
1369	2891.0	4900.77	435.41	944.78	81.75	5328.39
1521	2576.0	4906.45	424.98	945.91	79.91	5347.47
1681	2332.0	4961.32	419.72	956.52	79.02	5419.10
1849	2142.0	5061.81	419.04	975.92	78.99	5539.70
2025	1960.0	5129.15	415.35	988.91	78.37	5624.33
2209	1833.0	5279.17	419.46	1017.85	79.22	5798.34
2401	1665.0	5284.88	410.26	1018.95	77.56	5816.02
2601	1548.0	5382.73	409.60	1037.81	77.50	5933.44
2809	1428.0	5434.30	404.75	1047.75	76.64	6000.65
3025	1320.0	5484.97	399.86	1057.50	75.76	6066.86
3249	1248.0	5627.66	403.15	1085.01	76.44	6233.08
3481	1178.0	5755.23	404.99	1109.59	76.83	6383.01
3721	1080.0	5742.66	394.41	1107.13	74.87	6380.52
3969	1015.0	5835.26	393.04	1124.97	74.64	6492.55
4225	952.0	5912.41	390.23	1139.81	74.15	6587.85
4489	891.0	5974.14	386.00	1151.68	73.38	6666.44
4761	864.0	6190.77	394.99	1193.43	75.12	6914.09
5041	806.0	6226.52	388.30	1200.29	73.88	6964.63
5329	775.0	6396.05	392.91	1232.95	74.79	7161.30
5625	750.0	6592.62	399.64	1270.82	76.09	7387.71
5929	696.0	6589.67	389.32	1270.20	74.15	7396.40
6241	644.0	6572.36	377.72	1266.80	71.97	7389.48
6561	644.0	6907.88	395.61	1331.48	75.40	7768.36
6889	594.0	6866.96	381.78	1323.53	72.79	7735.92
7225	572.0	7022.91	384.26	1353.55	73.28	7918.92
7569	546.0	7136.06	383.02	1375.32	73.06	8055.30
7921	525.0	7280.76	384.22	1403.17	73.31	8226.40
8281	500.0	7379.07	381.41	1422.07	72.80	8346.93

THE FOLLOWING TIMES ARE BASED UPON A FITTED POLYNOMIAL OF 225 TERMS, WITH OPTIMAL ARRAY ALGEBRA PARAMETERS OF 15 AND 15.

NUMBER OF OBSERVATIONS	NUMBER OF POLYNOMIALS REQUIRED FOR ONE 1:50000 MAP SHEET	SECONDS OF L-S MULTIPLICATION TIME	SECONDS OF ARR-ALG MULTIPLICATION TIME	SECONDS OF L-S ADD TIME	SECONDS OF ARR-ALG ADD TIME	SECONDS SAVED
256	14148.0	4799.47	600.03	922.57	108.56	5013.45
289	12648.0	4856.59	588.33	933.95	106.86	5095.35
324	11446.0	4940.03	561.36	950.34	105.96	5203.06
361	10304.0	4970.49	569.17	956.49	104.06	5253.75
400	9328.0	5002.09	558.33	962.82	102.36	5304.22
441	8568.0	5080.51	553.87	978.14	101.80	5402.97
529	7161.0	5131.07	535.18	988.22	98.79	5485.32
625	6106.0	5208.70	522.13	1003.44	96.73	5593.28
729	5280.0	5295.94	512.00	1020.45	95.15	5709.24
841	4650.0	5423.40	507.38	1045.18	94.54	5866.66
961	4060.0	5463.69	495.08	1053.11	92.46	5929.45
1089	3564.0	5493.13	482.76	1058.84	90.34	5978.87
1225	3162.0	5542.15	473.20	1068.37	88.71	6048.61
1369	2891.0	5712.72	475.66	1101.32	89.31	6249.06
1521	2576.0	5727.72	463.93	1104.25	87.23	6280.80
1681	2332.0	5799.63	457.88	1118.15	86.21	6373.69
1849	2142.0	5924.36	456.85	1142.23	86.11	6523.62
2025	1960.0	6011.48	452.57	1159.04	85.40	6632.55
2209	1833.0	6193.96	456.79	1194.24	86.28	6845.14
2401	1665.0	6211.23	446.54	1197.57	84.42	6877.84
2601	1548.0	6334.71	445.62	1221.38	84.31	7026.17
2809	1428.0	6405.52	440.14	1235.02	83.34	7117.06
3025	1320.0	6475.74	434.63	1248.55	82.35	7207.31
3249	1248.0	6652.04	438.04	1282.53	83.05	7413.48
3481	1178.0	6811.43	439.87	1313.25	83.45	7601.37
3721	1080.0	6810.38	428.22	1313.01	81.28	7613.89
3969	1015.0	6930.53	426.59	1336.15	81.02	7759.08
4225	952.0	7033.37	423.40	1355.95	80.45	7885.48
4489	891.0	7118.96	418.68	1372.42	79.59	7993.10
4761	864.0	7382.82	428.31	1423.28	81.46	8296.33
5041	806.0	7439.48	420.94	1434.16	80.09	8372.62
5329	775.0	7650.16	425.83	1474.75	81.05	8618.04
5625	750.0	7892.33	433.00	1521.42	82.45	8898.30
5929	696.0	7905.90	421.72	1523.98	80.33	8927.84
6241	644.0	7903.44	409.06	1523.44	77.94	8939.88
6561	644.0	8306.45	428.33	1601.13	81.64	9397.61
6889	594.0	8277.78	413.28	1595.53	78.79	9381.24
7225	572.0	8475.54	415.88	1633.62	79.31	9613.97
7569	546.0	8624.63	414.45	1662.31	79.06	9793.43
7921	525.0	8810.37	415.67	1698.07	79.31	10013.46
8281	500.0	8943.31	412.56	1723.64	78.74	10175.66
8649	480.0	9116.97	412.40	1757.07	78.73	10382.91
9025	480.0	9511.42	429.07	1833.10	81.93	10833.53
9409	437.0	9395.39	406.12	1810.62	77.56	10722.33

THE FOLLOWING TIMES ARE BASED UPON A FITTED POLYNOMIAL OF 256 TERMS.
WITH OPTIMAL ARRAY ALGEBRA PARAMETERS OF 16 AND 16.

NUMBER OF OBSERVATIONS	NUMBER OF POLYNOMIALS REQUIRED FOR ONE 1:50000 MAP SHEET	SECONDS OF L-S MULTIPLICATION TIME	SECONDS OF ARR-ALG MULTIPLICATION TIME	SECONDS OF L-S ADD TIME	SECONDS OF ARR-ALG ADD TIME	SECONDS SAVED
289	12648.0	5559.86	647.17	1069.20	117.55	5864.35
324	11446.0	5657.17	638.91	1088.31	116.45	5990.13
361	10304.0	5694.33	624.98	1095.79	114.26	6050.89
400	9328.0	5732.93	612.57	1103.51	112.31	6111.56
441	8568.0	5824.94	607.21	1121.47	111.60	6227.59
529	7161.0	5888.44	585.88	1134.09	108.15	6328.50
625	6106.0	5983.28	570.86	1152.66	105.76	6459.32
729	5280.0	6089.59	559.14	1173.38	103.91	6599.93
841	4650.0	6242.21	553.51	1202.98	103.13	6788.56
961	4060.0	6296.42	539.57	1213.58	100.77	6869.66
1089	3564.0	6338.38	525.67	1221.78	98.37	6936.11
1225	3162.0	6403.38	514.84	1234.40	96.52	7026.42
1369	2891.0	6607.23	517.13	1273.77	97.10	7266.76
1521	2576.0	6634.60	504.03	1279.10	94.77	7314.90
1681	2332.0	6727.28	497.14	1297.01	93.60	7433.56
1849	2142.0	6880.59	495.72	1326.60	93.44	7618.04
2025	1960.0	6991.66	490.79	1348.04	92.61	7756.29
2209	1833.0	7211.77	495.11	1390.50	93.51	8013.64
2401	1665.0	7244.44	483.76	1396.80	91.45	8066.03
2601	1548.0	7398.51	482.53	1426.51	91.29	8251.19
2809	1428.0	7493.18	476.40	1444.75	90.20	8371.33
3025	1320.0	7587.74	470.24	1462.97	89.10	8491.37
3249	1248.0	7803.53	473.74	1504.57	89.82	8744.54
3481	1176.0	8000.62	475.54	1542.56	90.21	8977.42
3721	1080.0	8015.71	462.79	1545.43	87.84	9010.51
3969	1015.0	8169.24	460.87	1575.01	87.53	9195.85
4225	952.0	8303.62	457.28	1600.89	86.89	9360.34
4489	891.0	8418.91	452.05	1623.08	85.93	9504.01
4761	864.0	8737.58	462.31	1684.51	87.92	9871.85
5041	806.0	8821.05	454.23	1700.55	86.42	9980.96
5329	775.0	9080.30	459.39	1750.51	87.44	10283.99
5625	750.0	9375.92	467.01	1807.48	88.92	10627.46
5929	696.0	9411.99	454.73	1814.38	86.61	10685.02
6241	644.0	9430.35	440.98	1817.85	84.02	10723.19
6561	644.0	9910.61	461.66	1910.44	87.99	11271.39
6889	594.0	9900.24	445.34	1908.36	84.91	11278.35
7225	572.0	10148.05	448.05	1956.09	85.45	11570.64
7569	546.0	10341.02	446.43	1993.24	85.16	11802.68
7921	525.0	10576.24	447.65	2038.54	85.42	12081.71
8281	500.0	10751.92	444.23	2072.35	84.78	12295.26
8649	480.0	10974.47	443.98	2115.20	84.76	12560.93
9025	480.0	11448.78	461.85	2206.63	88.19	13105.38
9409	437.0	11342.82	437.08	2186.08	83.48	13008.35
9801	437.0	11812.72	453.98	2276.66	86.72	13548.67
10201	396.0	11681.92	427.02	2251.31	81.59	13424.63
10609	396.0	12146.59	442.91	2340.87	84.64	13959.91
10816	396.0	12382.34	450.97	2386.31	86.19	14231.50

APPENDIX G: EVALUATING THE FINAL POLYNOMIALS.

The Conventional Evaluation. The evaluation of the qth order polynomial

$$z = c_0+c_1x+c_2y+c_3x^2y+c_4xy^2+c_5x^2y^2+ \ . \ . \ . \ +c_mx^qy^q$$

has previously been accomplished by first creating two arrays, one of exponential values of x, another of y. Creating these arrays required $2(q-1)$ multiplications. The actual evaluation then required

$$\sum_{i=1}^{q} (2i + 1) \qquad \text{additions, and}$$

$$\sum_{i=1}^{q} (4i) \qquad \text{multiplications.}$$

Total adds and multiplications were therefore

$$q(q + 2) \qquad \text{additions, and}$$

$$2(q^2 + 2q - 1) \qquad \text{multiplications.}$$

This procedure was repeated for every desired elevation, normally 1.018 million times.

The Array Algebra Evaluation. The array algebra format, however, lends itself to a more efficient method of model equation evaluation. Consider the grid of polynomials in figure G1. Given these polynomials, we are required to produce elevation data at each "X". (Although figure G1 shows each polynomial describing $n' = 16$ elevation points, n' is not restricted to this value, nor is this 4 by 4 arrangement necessarily optimal). Now we define the local coordinate system to describe the relative positions of the elevation data within the polynomial p_{11} only. (see figure G2).

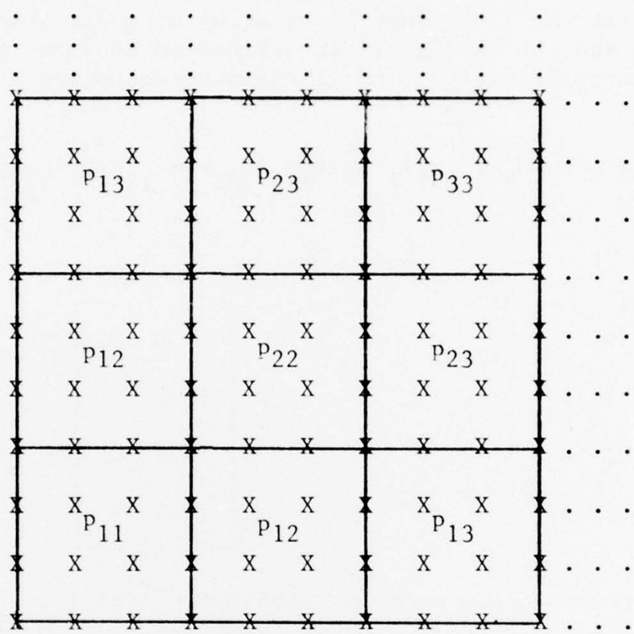


Figure G1. The Grid of Polynomials

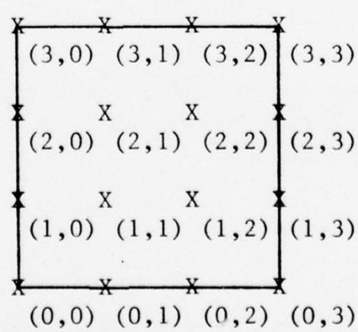


Figure G2. The Local Coordinate System

If the matrix X is defined as an array of x-direction values of the model equation and the matrix Y is defined as an array of y-direction values of the model equation, and if the model equation is defined as

$$z = c_{00}+c_{10}x_i+c_{20}x_i^2+c_{11}x_iy_i+c_{21}x_i^2y_i+c_{02}y_i^2+c_{12}x_iy_i^2+c_{22}x_i^2y_i^2$$

then

$$X = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \qquad Y = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}$$

The matrix of coefficients defining each polynomial is, of course,

$$C_A = \begin{bmatrix} c_{00} & c_{01} & c_{02} \\ c_{10} & c_{11} & c_{12} \\ c_{20} & c_{21} & c_{22} \end{bmatrix}$$

Note that the number of terms in the model equation (m') is chosen as 9, and C_A is an a' by b' matrix, where $a'b' = m$. The model equation can now be written in matrix notation as

$$Z_A = XC_AY^T$$

where Z_A is an e' by f' matrix of elevation points, and $e'f' = n'$.

A re-inspection of figure G1 shows that since many of the Z values lie on borders of two or more polynomials, not all Z values need be calculated for every polynomial.

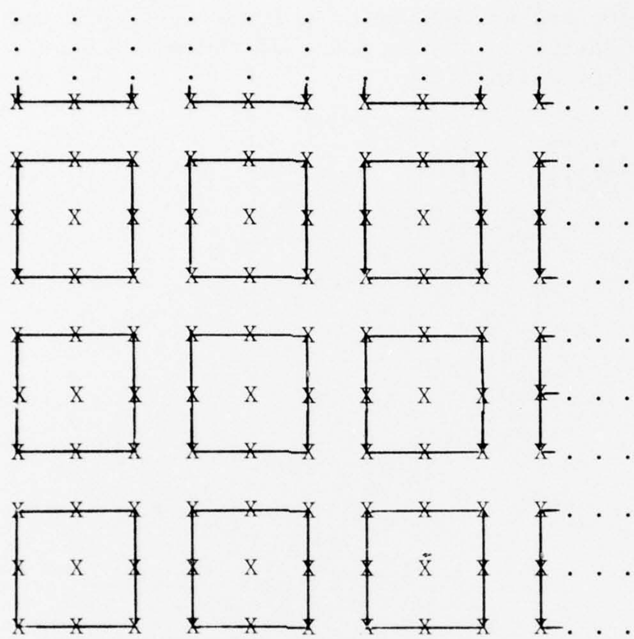


Figure G3. Evaluation of the Grid of Polynimials

Rather than e' by f' elevation points, only an $(e'-1)$ by $(f'-1)$ array (see figure G3) need be calculated. Also, an inspection of the X and Y matrices shows that they are independent of the coefficients of the model equation and dependent only upon the size of the x-y positional grid. Since each polynomial has an origin and a directional grid (figure G2) that is congruent to each origin and grid in the remaining polynomials, then X and Y (and of course Y^T) are constant for the entire map sheet and need be calculated only once. In a general case, the number of repetitive multiplications for $Z_A = XCY^T$ are

$$p' \left[(a')^2(e'-1) + b'(e'-1)^2\right]$$

and repetitive additions are

$$p' \left[a'(e'-1)(a'-1) + (e'-1)^2(b'-1)\right]$$

Mr. Jancaitis[8] describes the number of polynomials (p') as a function of the number of elevations on the sides of the grids upon which the final polynomials are defined:

$$p' = \left(\frac{1832}{e'+1} - 1\right)\left(\frac{2224}{f'+1} - 1\right)$$

[8]James R. Jancaitis, "Modeling and Contouring Irregular Surfaces Subject to constraints," USA Engineer Topographic Laboratories, Fort Belvoir, VA, ETL-CR-74-19, AD A010406, January 1975.